Lecture Notes on
APPLICATIONS OF JET PROPULSION
POWER PLANTS

H. S. Tsien 1951-1952

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I. The Atmosphere

We shall be mostly concerned with jet-propelled devices that go through air. It is thus of interest to acquaint ourselves first with the general characteristics of the atmosphere. This is the purpose of this lecture.

The atmosphere is the gaseous envelope of the earth which considered in a time interval of days and years can be taken to be unchanging; although considered in a time interval of the order of the life of the earth it is not. For our purpose then, the atmosphere maintains constant characteristics. These characteristics are not, however, absolutely constant, it varies with the days and the nights of the year. In fact the atmosphere is a state of dynamic equilibrium. In the aspect of composition, the atmosphere is constantly exchanging CO_2 and O_2 with the plant and the animal life; it also receives He through the radioactive decay of elements, and looses He by diffusion into the interstellar space. In the aspect of energy content, it receives solar energy by absorbing the short wave length radiation and it is heated by the warm earth. Therefore the atmosphere is really not static but has its many cycles and disturbances.

Atmospheric Regions

The earth's atmosphere is generally classified into three major regions - the troposphere, stratosphere and ionosphere, of which the latter two regions can be considered to constitute the upper atmosphere.

Lowest of these three atmospheric regions is the troposphere, which extends from sea-level to an average height of 7 miles (5 miles over poles, 10 miles over the equator). Here the stratosphere begins and reaches to a height of about 60 miles, from which the ionosphere, so called because it is a highly ionized or electrified region, extends through the remaining height of the atmosphere to the point where it merges with interplanetary space.

The troposphere is literally the disturbed and unsettled area of the earth's atmosphere. It is the region of cloud activity and weather; here are the storms, air masses, and warm or cold fronts which give the world its weather. The troposphere is the area in which dust, smoke, water vapor, and bacteria are largely found and here it is that the temperature drops at an average rate of about $1^{\circ}F$ per 300 feet of ascent.

The altitude at which this temperature decrease ceases and a constant temperature is reached is the demarcation line between the troposphere and stratosphere. The line is often referred to as the tropopause, or the dividing line between the two atmospheric regions. The altitude at which the tropopause is found varies considerably however. As pointed out, over the poles the troposphere extends only 5 miles high, but over the equator it reaches an altitude of 10 miles. In moderate latitudes it averages a height of 7 miles, despite seasonal variations (higher in summer and lower in winter). The tropopause temperature in middle latitudes (45°) is $-67^{\circ}F$; at the poles this temperature is higher than at the equator.

The stratosphere, extending from some 7 to 60 miles altitude, originally derived its name from the belief that it represented an orderly arrangement of wind strata or layers with an absence of turbulence and atmospheric mixing. This stratospheric concept of the past has recently had to be revised, as has the belief that the temperature remains constant throughout the height of the stratosphere.

That area of the upper air above 60 miles altitude, generally referred to as the ionosphere, is considered to extend to the outer limit of the atmosphere. The ionosphere is an area characterized by free electrons and by charged or electrified atoms and molecules of the unified upper air.

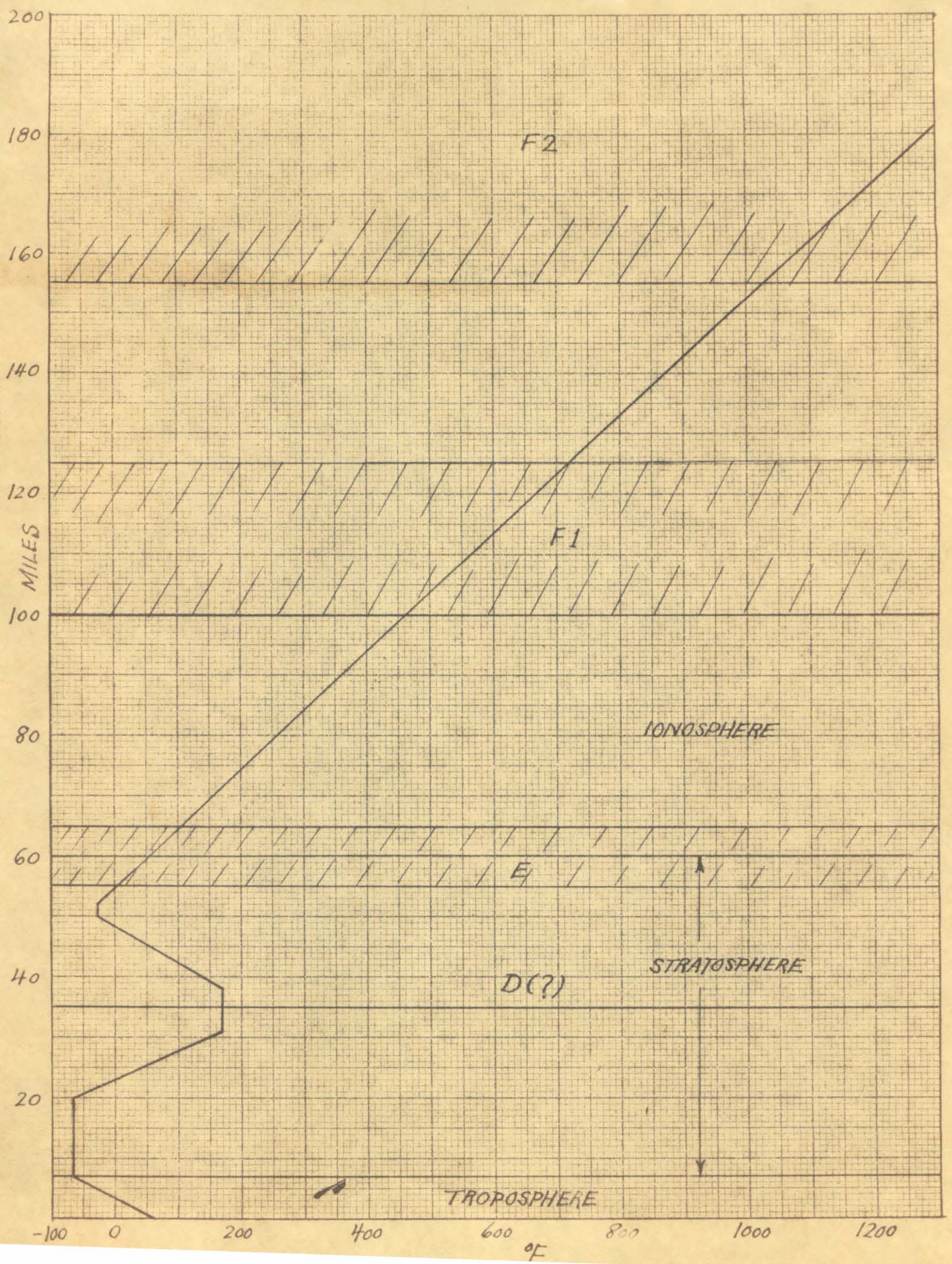
It has three major regions or layers where ion concentration is the greatest. Lowest and least ionized of these three layers is the E layer (55-65 miles). Next higher and more strongly ionized is the F1 layer (100-125 miles); highest and most intensely ionized is the F2 layer (155-215 miles). These heights are generalizations, because the layers vary greatly in thickness and altitudes. It is possible that the ionosphere additionally contains a G layer which, if it exists at all, is probably situated at a height between 250 and 435 miles. The G layer has been reportedly detected in regions near the geomagnetic equator.

There is still another ionized layer in the upper atmosphere. This is the D layer which is found variously at 35- to 60 miles heights. Although it is a region of ion concentration, it is only slightly ionized and of rather minor importance. Because of the relative unimportance of the D layer, the ionosphere is generally accepted as beginning with the much more significant E layer. Inasmuch as the E layer is usually found between 55 and 65 miles, this height is usually stated for base of the ionosphere.

The region where the ionosphere begins to merge with interplanetary space is often referred to as the exosphere, but the height to which the ionosphere actually extends is one of the many atmospheric unknowns.

Temperature Characteristics

As one ascends from sea-level up through the troposphere at a latitude of 45°, the temperature drops at a rate of 1°f every 300 ft. in altitude until a constant -67°F is reached at the tropopause or base of the stratosphere. Until recently it was widely believed that the temperature stayed at -67°F throughout the height of the stratosphere. The extention of man's knowledge of that region, however, has proved this belief to be in error.



F2
F1
IONOSPHERE
E
STRATOSPHERE
D(?)
TROPOSPHERE
MILES
°F

For a latitude of 45°, for instance, it is perfectly true that the temperature holds constant at -67°F, but only from 7 miles to 20 miles above which it undergoes a sharp increase. In the neighborhood of 31 miles, it measures 170°F and remains constant for the suceeding 7 miles. At 38 miles, a decrease in temperature sets in which drops the reading to a low of -27°F at about 50 miles. This temperature is held constant for perhaps 2 miles in altitude. At roughly 52 miles, however, the region is believed to undergo a severe and constant increase in temperature. This increase appears to occur at a constant or uniform rate, at least as high as the ionosphere's F2 layer where the altitude is about 200 miles, and the temperature is somewhere in the neighborhood of 1500°F. Above the F2 layer, it is possible that the temperature goes as high as that of interplanetary gas, estimated as being between 17,000°F and 26,000°F.

The upper air temperatures are the so-called kinetic temperatures defined according to the average energy contents of the molecules. They are derived or calculated on the basis of other atmospheric data (particularly pressures) and by such other indirect means as meteor observations, sound wave propagation, radio wave behavior, and spectroscopic analysis of the aurorae and of light from night sky (air glow). Needless to say these temperatures at high altitudes are only provisory, and subject to corrections as the upper air research progresses.

Atmospheric Composition

At sea-level, the earth's atmosphere is chemically composed of:

Nitrogen	78.08%	99.99%
Oxygen	20.95	
Argon	0.93	
Carbon dioxide	0.03	

Neon Helium Krypton Xenon Ozone Radone	0.01%

The most abundant components by far are obviously nitrogen and oxygen. There is little variation of this composition at low altitudes, presumably because the turbulence of the troposphere keeps these gases well and quite uniformly mixed.

With an increase in altitude, there is somewhere a decrease in atmospheric mixing by turbulence and vertical currents. As a consequence, the gaseous components of the air might be expected to rearrange themselves according to weight; a diffusive separation of these atmospheric constituents should occur, with the heavier gases tending to separate and settle. The height at which such diffusive separation does begin, however, is one of the unsolved mysteries of the upper atmosphere. Some authoritative estimates place the altitude as high as 250 km. (115 miles).

Oxygen and nitrogen, the dominant constituents of the earth's atmosphere, undergo changes as altitude is increased. While the ratio of nitrogen to oxygen at sea-level is 78% to 21% of the entire gaseous composition of the air at that height, it is possible that this ratio undergoes a change in the neighborhood of 30 miles above sea level, where it becomes 82% nitrogen and 18% oxygen in a latitude of 45°.

Nitrogen and oxygen show a variation in their abundance because they undergo basic chemical changes. They are normally molecules but at upper atmosphere, O_2 is converted to O_3 and O and N_2 to N.

The increase in temperature of the stratosphere at about 20 miles in 45° latitude is no accident; it is the direct result of the presence of ozone at that atmospheric height. Ozone is produced by the effect of the sun's ultrviolet radiation upon O_2 and it has the property of absorbing heat from solar radiation.

Ozone has been detected in varying quantities from sea-level to heights in excess of 30 miles; its heaviest concentration appears to be located at an altitude of about 15 miles. Although it extends through 30 miles or more, its actual concentration is relatively small. Were the earth's ozone reduced to standard pressure and temperature, it would form a layer only 1/10" thick. But it's sparsity should not cause us to overlook the importance of ozone.

It is widely believed that the dissociation of oxygen from its molecular form to its atomic form begins in the region from 50 to 80 miles and possibly continues up through the remainder of the atmosphere, and that nitrogen undergoes similar dissociation from N_2 to N. Atomic nitrogen has been observed in aurorae, but little is generally known about the dissociation of N_2 in the upper air. Regardless, however, of whether they exist in molecular or atomic states, nitrogen and oxygen are the main chemical constituents of the atmosphere at least up to the altitudes of the highest observed aurorae (700 miles).

The spectroscopic analysis of light emissions at heights between 45 and 450 miles, shows the presence of sodium at such altitudes. The origin of this sodium is not known, but it is sometimes attributed to meteoric or volcanic dust.

While it was not originally expected to find sodium in the upper air, many physicist did expect to detect large quantities of He. It was reasoned that, because of radioactivity and because of the constant liberation of helium into the atmosphere in natural gases emerging from the earth's interior, He would be naturally abundent in the upper air. Because of its lightness, He was further assumed to rise to the upper part of the atmosphere which, according to many scientist, would be constituted almost entirely of He in company with a still lighter gas, H. To date, however, man has no concrete evidence, spectroscopic or otherwise, that this atmospheric ceiling of He and H exists. A possible explanation is that He rises to the top of the ionosphere and then escapes from the earth's atmosphere by entering interplanetary space. This is all the more likely due to the high temperature existing at high altitude.

The amount of water vapor in the air is a highly variable factor, as the daily reading of the relative humidity in the troposphere will verify. However it is largely concentrated in the troposphere. There is water vapor in the regions of upper air, but its extent and location are subjects of differences of opinion. At least one balloon flight, using a specially developed electronic dew point hygrometer, has indicated the presence of a thin saturated layer in the lower stratosphere at a height of about 10 miles. Water vapor has also been spectroscopically detected in the upper stratosphere. The existence of nacreous and noctilucent clouds lends additional support to the belief that water vapor is contained in the upper air.

Force Balance-Pressure

If p is the pressure at an altitude y, and   is the mass density at y and g is the gravitational acceleration, the equilibrium of vertical forces requires

$$dp = -g\rho\, dy \qquad (1.1)$$

The value of g can be approximated as

$$g = g_0 r^2/(r+y)^2 \qquad (1.2)$$

where g_0 is the gravitational acceleration at the surface and r is the radius of the earth. Assuming perfect gas law to hold and let R be the universal gas constant,

$$\rho = \frac{p\, M(y)}{R\, T(y)} \qquad (1.3)$$

where M(y) is the average molecular weight of the atmosphere at y and T(y) is the absolute temperature at y. Therefore if p_0 is the ground pressure,

$$\log(p_0/p) = \int_0^y \frac{g_0 r^2}{(r+y)^2}\,\frac{M(y)\,dy}{R\,T(y)} = \frac{g_0}{R}\int_0^y \frac{M(y)\,dy}{(1+\frac{y}{r})^2\,T(y)} \qquad (1.4)$$

(1.4) shows that the pressure and hence the density of the atmosphere at altitudes can be calculated if the temperature and the composition of the air at altitudes are known. The relation can be also used to check the consistency of measured data.

Two things are to be noticed: For an isothermal atmosphere of constant composition,

$$\log(p_0/p) = \frac{g_0 r M}{RT}\int_0^y \frac{d(\frac{y}{r})}{(1+\frac{y}{r})^2} = \frac{g_0 r M}{RT}\left[1 - \frac{1}{1+\frac{y}{r}}\right] \qquad (1.5)$$

Therefore as $y \rightarrow \infty$, $\log(p_0/p)_\infty \rightarrow \frac{g_0 r M}{RT} = \frac{\gamma g_0 r}{a^2}$

where γ is the ratio of specific heats and a is the velocity of sound. The interesting point is that the pressure p at $y = \infty$ is not zero but finite, although quite small.

Secondly, the behavior of pressure at extreme altitude is really quite general, since both composition and temperature do settle down to constant values (values for interplanetary gas) and the pressure of interplanetary gas can be estimated. This fact may be considered as to indicate that the gravitational attraction is not strong enough to condense all matters into heavenly bodies.

The laborious calculation using temperature and composition data indicated in the previous sections of this lecture are carried out. The results tabulated. (See, "Handbook of Supersonic Aerodynamics" Vol. I. NAVORD Report 1488, (1950)). The following is the abridged values for 40° latitude.

Altitude Ft.	Weight lb/ft ^{3}	Pressure psi	Pressure "Hg
Sea Level	0.07651	14.70	29.92
10,000	0.05649	10.11	20.58
20,000	0.04075	6.76	13.75
30,000	0.02861	4.36	8.88
40,000	0.01872	2.72	5.54
50,000	0.01161	1.69	3.44
60,000	0.00720	1.05	2.13
70,000	0.00447	0.649	1.32
80,000	0.00277	0.403	0.820
90,000	0.00172	0.250	0.509
100,000	0.00107	0.155	0.315

Besides the force balance, we may also try to compute the energy balance. Here the difficulty is however that the transport processes are complicated by the turbulent motion of the air mass, which as stated in the previous section, extends to very high altitude.

Upper Air Winds

Winds in the stratosphere at balloon heights are now known to experience a seasonal change, flowing west to east in winter and east to west in summer; greater wind speeds predominate in the winter. Tracking of noctilucent clouds indicates that winds at their altitude (50 miles) are as great as 400 miles per hour and generally flow from the northeast. Wind values derived from meteor trail studies has shown considerable variation; in the daytime, meteor trains drift towards the east, at night towards the west. Below a height of 50 miles, they drift toward the west, but above 50 miles a drift toward the east predominates. Wind speed, derived from meteor sightings, average about 80 mph at a height of 27 miles and 120 mph at 54 miles. At heights of about 60 to 75 miles over the northeastern United States, sporadic E layer observations have shown wind velocities of 60 mph, 90 mph, and as high as 290 mph, all from east to west.

The existence of turbulence and vertical air movement in the upper atmosphere is now believed to be a well-established fact. For one thing, the composition of the atmosphere has been found to be approximately the same at 50 miles as in the troposphere; this indicates that there is mixing and stirring of the air up to as high as 50 miles. Some authorities are willing to place the altitude for the begining of diffusive equilibrium as high as 155 miles. Other evidence of upper air turbulence can be found in the behavior of noctilucent clouds, meteor trails and sounding rockets. Noctilucent clouds are often wavy in appearance and continually change their shape in a manner that would be logically caused by vertical air movement. The straight lines of meteor trails are frequently rapidly distorted and broken up as if torn apart by very severe atmospheric turbulence.

Pressure readings on more than one high altitude rocket have indicated that the rocket might have been in a strong atmospheric updraft. All this leads to the conclusion that the upper air, primarily the stratosphere, is a region of turbulence and strong vertical as well as strong horizontal winds.

2. Aerodynamic Problems

A jet-propelled device can be made to travel through the atmosphere from sea-level up to the highest altitude and with speeds up to 10,000 mph. The immediate question in the design of such devices is the aerodynamic forces which the device will experience. For many problems, we are only interested in the drag, but for many others we are particularly interested in the lift of the device to obtain long glide. We are always of course interested in the control and stability problem. To supply this design information is the aerodynamic problem.

It is general practice to express the forces and the moments in non-dimensional coefficient forms. These coefficients are obtained by dividing the actual force and moment by the dynamic pressure $\frac{1}{2}\rho U^2$ (ρ the air density, U the flight speed) and a proper power of the typical dimension of the body. These coefficients are functions of two characteristic numbers: the Reynolds number Re and the Mach number M. The aerodynamic problem then is really to find the functional relations between the coefficients to the Reynolds number and the Mach number.

Realms of Fluid Mechanics

As a first approximation, the gas may be considered as an aggregate of rapidly moving particles which are constantly colliding with each other.

Then the coarseness of the structure of the gaseous medium can be expressed by the parameter ℓ which is the distance the particles or molecules travel between collisions. Since the instantaneous velocity distribution and density distribution in the gas are far from uniform, one can only conveniently use the statistical average of the quantity instead of the instantaneous value of the quantity. The distance ℓ is then the statistical average over the billions and billions of molecules concerned. This average ℓ is called the mean free path of the gas. If the mean free path is small in comparison with the dimension of the flow field or the dimension of the body in the flow field, then the gas can be considered as a continuem and the ordinary gas dynamics is sufficient for the analysis of these flows. If the mean free path is not negligible when compared with the dimension of the body, then the effects of the discrete character of the gas must be taken into account in the calculation.

The mean free path ℓ can be calculated from the mean velocity of the molecules, the density of the gas and the "effective radius of the molecules." However, there is no direct way of measuring the effective radius of molecules; therefore it is more fruitful to express the quantity ℓ in terms of a measurable quantity such as the viscosity of gas. This is easily done by the following consideration. If the gas flows in the x-direction with the macroscopic velocity u/y) the gradient is then $\frac{\partial u}{\partial y}$. Now the gas molecules in a lower layer move into an upper layer with the average velocity $\bar{v}$ of the molecules. The distance they will travel before they lose their identity by collision with other molecules is ℓ. The difference in macroscopic velocity of the layer where the molecules originated and the layer where the molecules are mixed is then $\ell \left(\frac{\partial u}{\partial y}\right)$. The mass of molecules crossing a unit area of the layer is proportional to $\rho \bar{v}$ where ρ is the density of the fluid. Therefore, the momentum exchange between layers is proportional to $\rho \bar{v} l \left(\frac{\partial u}{\partial y}\right)$. This is the viscous shearing stress τ. Since the coefficient of viscosity is defined as

$$\tau = \mu \left(\frac{\partial u}{\partial y}\right)$$

μ is proportional to $\rho \bar{v} l$. The calculation of this coefficient of proportinality is however not easy, according to the Kinetic theory, it is

$$\mu = 0.499 \rho \bar{v} l \tag{2.1}$$

The average velocity $\bar{v}$ is closely connected with the "velocity of sound" a by the equation

$$\bar{v} = \sqrt{\frac{1}{\pi \gamma}}\, a \tag{2.2}$$

where γ is the ratio of specific heats. Thus the mean free path is given in terms of the kineanatic viscosity $\nu = \frac{\mu}{\rho}$ and the velocity of sound a by

$$l = 1.255 \sqrt{\gamma} \left(\frac{\nu}{a}\right) \tag{2.3}$$

The flow conditions near the wall for the case in which the mean free path l is small but not negligible compared with the body dimension L or the thickness S of the boundary layer were first investigated theoretically by J. C. Maxwell in 1879. It is found that the flow velocity at the wall is no longer zero as is the case in ordinary fluid flow but it is

$$u = \frac{\mu}{\beta} \frac{\partial u}{\partial y} + \frac{3}{4} \frac{\mu}{\rho T} \frac{\partial T}{\partial x} \qquad at \quad y = 0 \tag{2.4}$$

The ratio of μ and β is given by

$$\frac{\mu}{\beta} = 0.998 \left[(2/f) - 1\right] l \tag{2.5}$$

f is the fraction of the tangential momentum of the impinging molecules transmitted to the wall. Maxwell himself interpreted the fractional value of f as meaning that a fraction f of the surface reflects diffusely and the remainder specularly. f is generally less but close to 1.

Similar to this velocity discontinuity, there is a temperature discontinuity at the wall first found by Smoluchowski in 1898. Namely,

$$\kappa(T - T_w) = \lambda\left(\frac{\partial T}{\partial y}\right) \quad , \quad y = 0 \tag{2.6}$$

where T is the temperature of the gas at the wall, Tw is the temperature of the wall and κ is the coefficient of temperature discontinuity. λ is, of course, the coefficient of heat conduction.

$$\frac{1}{\kappa} = 1.996\left(\frac{2-\alpha}{\alpha}\right)\frac{\gamma}{\gamma+1}\,\frac{1}{\mu c_p}\,\ell \tag{2.7}$$

where α is the accommodation coefficient introduced by Knudsen. α can be defined as the fractional extent to which those molecules that fall on the surface and are reflected or re-emitted from it, have their mean energy adjusted or "accommodated" toward what would be if the returning molecules were "at temperature" of the wall.

Such flow with small $\frac{\ell}{L}$ or $\frac{\ell}{\delta}$ ratios is thus appropriately called as the slip flow. To connect the criterion $\frac{\ell}{L}$ or $\frac{\ell}{\delta}$ to the more familiar characteristic parameters of flow, we note

$$\frac{\ell}{\delta} = \frac{L}{\delta}\cdot\frac{\ell}{L} \sim \frac{L}{\delta}\cdot\frac{\nu/a}{L} \sim \frac{L}{\delta}\cdot\frac{\nu}{UL}\cdot\frac{U}{a} \sim \frac{L}{\delta}\,\frac{M}{Re}$$

For small Reynolds number $\frac{L}{\delta} \sim 1$, then

$$\frac{\ell}{\delta} \sim \frac{M}{Re} \quad , \quad Re \sim 1 \tag{2.8}$$

For large Reynolds numbers $\frac{L}{\delta} \sim \sqrt{Re}$, then

$$\frac{\ell}{\delta} \sim \frac{M}{\sqrt{Re}} \quad , \quad Re \gg 1 \tag{2.9}$$

Eqs. (2.8) and (2.9) then connected the slip flow criteria with the Reynolds number and the Mach number. A concept first developed by Th. von Kármán.*

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* Th. von Kármán "Gas-theoretische Deutung der Reynoldschen Kennzahl" ZaMM 3 (1923)

If the mean free path is much larger compared with the body dimensions, one enters an entirely new realm of fluid mechanics. Here the chances for the collision of molecules among themselves are much smaller than the chances for the collision of molecules with the wall or the surface of the body. Therefore, for the calculation of forces, one need only consider the impact of a stream of molecules with a velocity and energy distribution determined by the thermal equilibrium in the free stream, the Maxwellian distribution. The greatest simplification comes from the fact that one need not consider the distortion of the Maxwellian distribution due to collision of the re-emitted molecules with the molecules in the stream. This realm of fluid flow is called free molecule flow.

Therefore we set up the somewhat arbitrary criteria that for $\frac{\ell}{\delta} < \frac{1}{100}$, we have the ordinary gas dynamics, for $\frac{\ell}{\delta} > \frac{1}{100}$ we have slip flow, and finally for $\frac{\ell}{L} > 10$, we have the free molecule flow. The whole plane can then be divided into regions, each for one type of fluid motion.

Gas Dynamics

The dominant concept in the fluid mechanics of the gas dynamics realm is the boundary layer, due to L. Prandlt. Through the boundary layer concept, we are able to separate viscous effects from the dynamic effets. The dynamic effects are considered first in the so-called perfect fluid theory. In this part of gas dynamics then the aerodynamic coefficients are functions of Mach number only. In fact the flow is called subsonic, when the Mach number is less than one so that in no part of the field is the local Mach number equal to 1. When the Mach number≈ 1, the flow is called transonic. When $M > 1$, the flow is supersonic. When $M \gg 1$, the flow is hypersonic. For aerodynamicist, the point of particular interest is then relation between the aerodynamic coefficients and the Mach number. In this problem, the engineer is helped by another powerful concept: this is the idea that if the body moving through the air is thin or slender the distrubance created by the body would be small. The concept of small distrubance led immediately to the linearized theories in subsonic and supersonic flows and to a greatly simplified theory on transonic and hypersonic flows. Morevore, the most rewarding consequence of the small distrubance concept is the very useful similarity rules:

a) Subsonic Flows - Prandtl - Glauert rules $(1/\sqrt{1-M^2})$
b) Transonic Flows - von Kármán Similarity rules
c) Supersonic Flows - Ackeret rules $(1/\sqrt{M^2-1})$
d) Hypersonic Flows - Hypersonic Similarity rules

It is almost certain that these powerful tools together with windtunnel research will solve all problems of this type. Therefore it is perhaps not too much to say that gas dynamic flows of perfect gas is essentially solved and only details need to be filled out.

Not so is however the boundary layer problem. Here the viscosity and heat condition effects are the predominating ones. The dynamic pressure propagation takes only a secondary position. Therefore the character of the boundary layer is not greatly changed by increasing the Mach number of the free stream from subsonic to supersonic ones. The theory of laminar boundary layer and the problem of the stability of the laminar layer are satisfactorily solved, except the very high speed cases where dissociation of molecules has to be taken into account. The unsolved problem is the problem of turbulent boundary layer. Here the theoretical understanding will be perhaps very difficult, but a great amount of experimental data certainly helps. Fortunately we do not expect a radical change in turbulent boundary layer behavior from subsonic to supersonic flows. A very intriging case is the turbulent boundary layer of hypersonic flow where the turbulent fluctuating velocities may be of the same order of magnitude as the velocity of sound.

Slip Flows

It is physically clear that slip of fluid over the surface of the body will reduce the friction and the temperature jump over the body will reduce the heat flux. The greatest difficulty in this field is the uncertainty about what basic equations to use. Recent test data from the University of California indicate, however, if the Mach number is moderate ($3 < M$), theoretical results from the combination of Navier-Stokes equations and the slip conditions agree well with facts. Of course, this does not say that Navier-Stokes equations can be adequate for higher Mach numbers.

Two facts are worth pointing out: In ordinary boundary layer flow, the influence of the so-called "second viscosity coefficient" or bulk viscosity is not felt because the main effect is the shear, not the volume change. In slip flows, the Reynolds number may be so small as to make the boundary layer concept invalid. Then the effect of bulk viscosity can be of importance.

Secondly, the Berkeley test results show that the temperature recovery factor is greater than unity. This is a fact which may be of considerable importance in engineering design.

Free Molecule Flow

Here the "viscous" effects completely predominate the dynamic effects. Therefore the Mach number changes never introduced any startling phenemena. There is no change in characteristics of flow in going from subsonic to supersonic to hypersonic. Since the flow is controlled by the wall re-emission of molecules, the law of interaction between the molecules and the wall is of primary importance here. Unfortunately, very little is known on this subject. The necessary basic knowledge here is the molecular distribution function of the re-emitted molecules for a beam of uniform impinging molecules of specified energy and velocity at a specified wall temperature. At very high speeds, such as a falling meteor, it will also be necessary to consider dissociation and chemical reacting i.e., not only is the energy state of the molecule changed by collision with the wall, but also its structure.

It is generally found that the value of f is very close to 1, but is small. The explaination here is that f is close to 1 because of the diffraction of molecular beam by the crystal lattice of the solid surface and the multicrystalline structure of engineering materials. Actually the molecule spends very little time on the solid to get accommodated to it.

3. Vertical Trajectory of Rockets

The simplest performance problem of the dynamic trajectory of a jet-propelled device is the vertical ascent of rockets. It is obvious that vertical trajectory is the trajectory for sounding rockets.* But even for long-range rockets the initial powered flight is often vertical for reasons of convenience in launching, minimum drag loss, etc. Launching is made easier by the elimination of launching ramp, and drag loss is smaller by penetrating the dense atmosphere at the maximum rate. Therefore the study of vertical trajectory is an important problem besides being mathematically simple.

General Relations

If m is the instantaneous mass of the rocket at time t, y the altitude, c the effective exhaust velocity, D the drag and g the gravitational attraction, then

$$c\frac{dm}{dt} + m\left(\frac{d\dot{y}}{dt} + g\right) = -D(\dot{y}.y) \qquad (3.1)$$

c and g are not exactly constants; c varies because of the reduction of atmospheric pressure with altitude and g reduces with height. However these variations may be neglected without seriously influencing our results. Then (3.1) can be written as

$$e^{-\frac{\dot{y}+gt}{c}} \frac{d}{dt}\left(e^{\frac{\dot{y}+gt}{c}} m\right) = -\frac{D}{c}$$

thus

$$m = e^{-\frac{\dot{y}+gt}{c}}\left[\int_0 -\frac{D}{c}\, e^{\frac{\dot{y}+gt}{c}}\, dt + \text{Constant}\right]$$

If m_1 is the mass of the rocket at burn-out, at $t=t_1$, and $\dot{y} = \dot{y}_1$, then

$$m_1 = e^{-\frac{\dot{y}_1+gt_1}{c}}\left[\int_0^{t_1} -\frac{D}{c}\, e^{\frac{\dot{y}+gt}{c}}\, dt + \text{Constant}\right]$$

Hence

$$m = e^{-\frac{\dot{y}+gt}{c}}\left[-\int_0 \frac{D}{c}\, e^{\frac{\dot{y}+gt}{c}}\, dt + m_1 e^{\frac{\dot{y}_1+gt_1}{c}} + \int_0^{t_1} \frac{D}{c} e^{\frac{\dot{y}+gt}{c}}\, dt\right] \qquad (3.2)$$

Let m_0 be the initial mass of the rocket at $t = 0$ when the velocity is $\dot{y}_0$, then

$$m_0 = e^{-\frac{\dot{y}_0}{c}}\left[\int_0^{t_1} \frac{D}{c}\, e^{\frac{\dot{y}+gt}{c}}\, dt + m_1 e^{\frac{\dot{y}_1+gt_1}{c}}\right] \qquad (3.3)$$

or

$$\frac{m_1}{m_0} = e^{-\left[\frac{(\dot{y}_1-\dot{y}_0)+gt_1}{c}\right]} - e^{-\left[\frac{\dot{y}_1+gt_1}{c}\right]} \int_0^{t_1} \frac{D}{c\,m_0}\, e^{\frac{\dot{y}+gt}{c}}\, dt \qquad (3.4)$$

(3.4) immediately gives two important results: Firstly, since D is proportional to the cross-sectional area of the rocket and hence is proportional to d^2, d the diameter of the rocket, while m_0 is proportional to the volume of the rocket or d^3, the importance of drag decreases with increase in the size of rocket. For large rockets then, the drag influence can be computed as a small perturbation of the trajectory without drag.

Secondly, when drag is negligible,

$$\frac{m_1}{m_0} = e^{-\left[\frac{\dot{y}_1 - \dot{y}_0 + g t_1}{c}\right]} \quad , \quad (D = 0) \tag{3.5}$$

Therefore for a given initial velocity and final velocity, the ratio of initial mass to the final mass is the least, or the expenditure of propellant is the smallest, if $t_1 = 0$, i.e., use the propellant all at once as an impulse. If t_1 is fixed, then the mass ratio $(D = 0)$ is quite independent of the manner acceleration and thrust varies with time and altitude. (3.5) can be also written as

$$\dot{y}_1 - \dot{y}_0 = c \log\left(\frac{m_0}{m_1}\right) - g t_1 \tag{3.6}$$

Thus the loss in final velocity due to the action of gravity is $g t_1$.

2. Constant Thrust vs. Constant Acceleration *

For long range rockets or rockets of high altitudes, the altitude at burn-out is generally a small part of maximum altitude. Therefore the performance of the rocket can be approximately determined by the maximum velocity $\dot{y}_1$ at burn-out alone. In this section, we shall make two more simplifying assumptions: the initial velocity $\dot{y}_0 = 0$ and $D = 0$.

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* Cf. Malina, F. J. and Summerfield, M., "The Problem of Escape from the Earth by Rocket". J. Aero. Sciences 14:471-480 (1947)

Ehricke, K. A., "A comparison of Rocket Propulsion at Constant Thrust and at Constant Acceleration", J. Detroit Rocket Soc., 5:50-63 (1951)

Then the general relation between mass ratio and speed is

$$\dot{y}_1 = c \log \frac{1}{1-\varsigma} - g t_1 \tag{3.7}$$

where ς is the propellant loading ratio defined by

$$\varsigma = \frac{m_0 - m_1}{m_0} \tag{3.8}$$

It is the ratio of propellant weight to the gross weight. For V-2, ς is 0.75.

Now if the thrust is constant, then the maximum acceleration is

$$c\, \frac{m_0 - m_1}{t_1}\, \frac{1}{m_1} - g = a_{max} = n_{max}\, g \tag{3.9}$$

where $\overset{n}{\cancel{N}}_{max}$ is the value of acceleration in g. Thus (3.7) can be written as

$$\dot{y}_1 = -c\left[\log(1-\varsigma) + \frac{\varsigma}{1-\varsigma}\, \frac{1}{(n_{max}+1)}\right] \tag{3.10}$$

This relation has the interesting property that for a fixed value of ${}^{n}N_{max}$, determined by the physical limitations of the pay load, there is a maximum $\dot{y}_1$. This is determined by

$$\frac{\partial \dot{y}_1}{\partial \varsigma} = 0 = \frac{1}{1-\varsigma^*} - \frac{1}{(n_{max}+1)}\left[\frac{\varsigma^*}{(1-\varsigma^*)^2} + \frac{1}{1-\varsigma^*}\right]$$

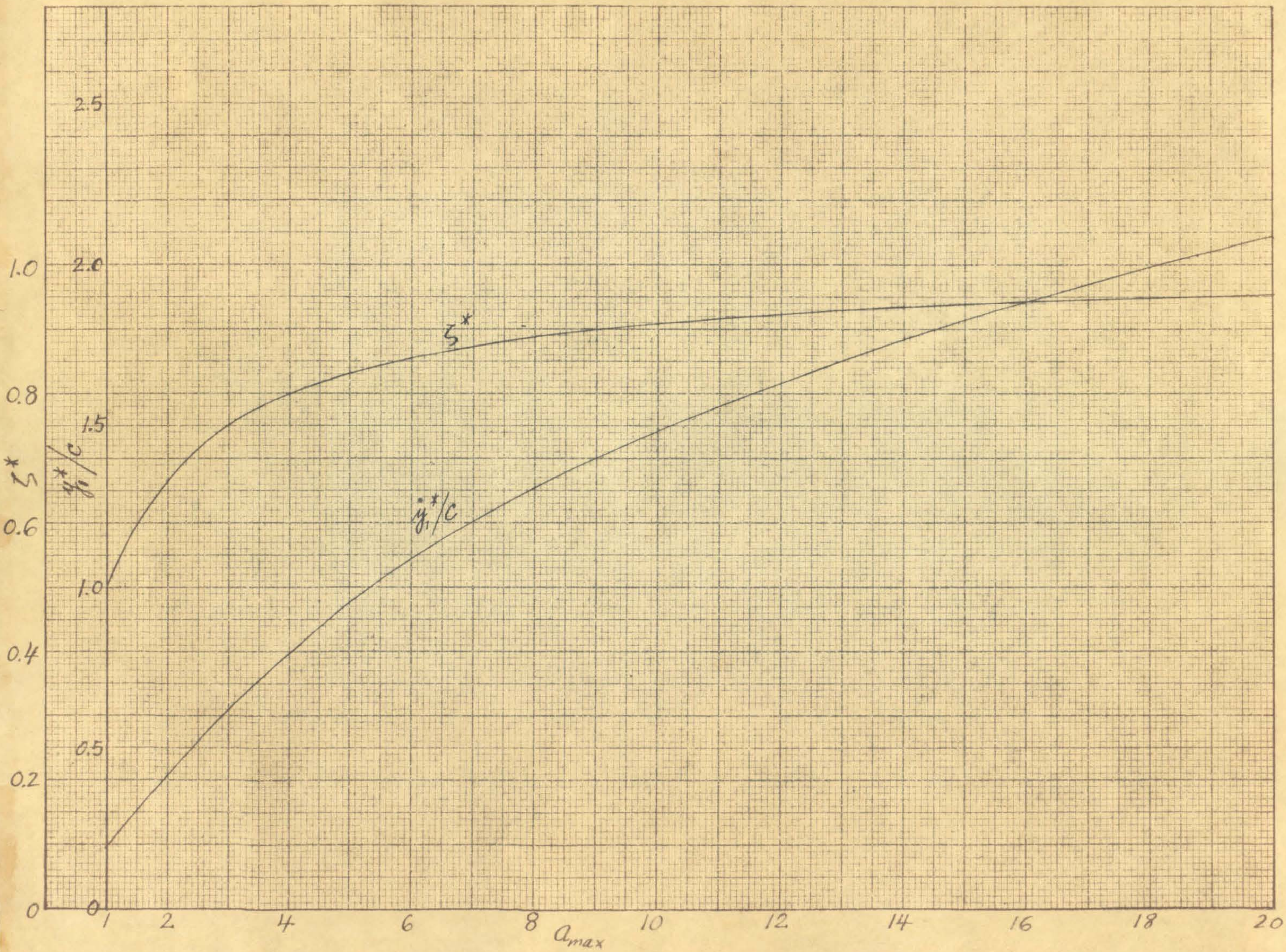
where ς^* is the value of ς at maximum $\dot{y}_1$. Solving for ς^*, we have

$$\varsigma^* = \frac{n_{max}}{n_{max}+1}, \qquad t_1^* = \left(\frac{c}{g}\right)\left(\frac{n_{max}}{n_{max}+1}\right) \tag{3.11}$$

Therefore the maximum $\dot{y}_1^*$ corresponding to ς^* is

$$\dot{y}_1^* = c\left[\log(n_{max}+1) - \frac{n_{max}}{n_{max}+1}\right] \tag{3.12}$$

The physical ~~relation~~ reason for this limitation on maximum speed is of course the low acceleration and the inefficient way of utilizing propellant at the start.



0
0.2
0.4
0.6
0.8
1.0
0.5
1.0
1.5
2.0
2.5
ζ^*
$\dot{y}_1^*/c$
1
2
4
6
8
10
12
14
16
18
20
a_{max}

If the thrust is varying, but the <u>acceleration is constant</u>, then

$$gt_1 = \frac{\dot{y}_1}{n_{max}} \tag{3.13}$$

and

$$\dot{y}_1 = -c\,\frac{n_{max}}{n_{max}+1}\log(1-\varsigma)\ ,\ t_1 = -\frac{c}{g}\,\frac{1}{n_{max}+1}\log(1-\varsigma). \tag{3.14}$$

To compare these two cases, let us take $n_{max} = 5$. Then for constant thrust, the maximum velocity is reached when $\varsigma^* = 5/6 = 0.833$, and $\dot{y}_1^*/c = 0.959$. For constant acceleration of $N_{max} = 5$, the same maximum speed is possible with a smaller loading ratio $\varsigma = 0.684$. Of course the actual difference is not so great because the constant thrust rocket has a larger burning time and hence greater altitude at burn-out. But nevertheless, the constant acceleration rocket has a better performance if the limitation is on the maximum acceleration the payload can withstand. Hence for practical design, some degree of thrust programming is indicated.

<u>Prob. 3.1</u>

Assume constant g and $\dot{y}_0 = 0$, show that the vacuum summit altitude Y_1 for a constant thrust rocket of maximum acceleration $N_{max}\,g$ is

$$Y_1 = \frac{c^2}{g}\left[\frac{1}{2}\log^2\frac{1}{1-\varsigma} - \frac{\log\frac{1}{1-\varsigma} - \varsigma}{(1-\varsigma)(n_{max}+1)}\right] \tag{3.15}$$

For a constant acceleration rocket of acceleration $N_{max}\,g$, the vaccum summit altitude Y_2 is

$$Y_2 = \frac{c^2}{g}\,\frac{n_{max}}{2(n_{max}+1)}\log^2\frac{1}{1-\varsigma} \tag{3.16}$$

<u>Prob. 3.2</u>

For the constant thrust rocket, let $\varsigma = 5/6$, $n_{max} = 5$, what is the value of $Y_1/\frac{c^2}{g}$? If the constant acceleration rocket is made to have the same value of $Y_2/\frac{c^2}{g}$, at same N_{max}, what is the required ζ ?

Answer:

$Y_1/(c^2/g) = 0.64 \qquad \zeta = 0.711$

Drag Correlation

The drag of a rocket in vertical flight is a function of altitude and velocity. This functional relation is rather complicated. Thus in general the integration of the equation of motion of rocket including drag cannot be done analytically and numerical methods are necessary. If the calculation is to be done by desk calculator manually, any one of the numerical integration methods for ordinary differential equation can be used.*

However, if the size of the rocket is very large, say a rocket of 50 tons, or if the rocket is the later step of a step rocket so that the air density is very small, the drag is very small in comparison with inertia force. Then as a zeroth order approximation, the drag effects will be neglected and the performance is calculated as if the atmosphere does not exist. This is done in the previous section. After the zeroth approximation is computed, the drag effect is calculated as a correction.

In general, if F is the thrust

$$m\frac{d\dot{y}}{dt} = F - mg - D \tag{3.17}$$

Now let the magnitude of the drag be represented by D_0 so that D/D_0 is a non-dimensional quantity of order one. Then

$$\frac{m}{m_0 g}\frac{d\dot{y}}{dt} = \frac{F}{m_0 g} - \frac{m}{m_0} - \frac{D_0}{m_0 g}\frac{D}{D_0} \tag{3.18}$$

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* Cf. Whittaker and Robinson, "Calculens of Observation"

If the size of the rocket is very large, then $D_0/M_0 g$ will be quite small. Therefore $D_0/m_0 g$ is non-dimensional parameter to measure the drag effect. Then we can write y as a power series in this parameter.

$$y = y^{(0)} + \frac{D_0}{m_0 g} y^{(1)} + \text{--------} \tag{3.19}$$

By substituting (3.19) into (3.18) and equating equal powers of $D_0/m_0 g$,

$$\frac{m}{m_0 g} \frac{d\dot{y}^{(0)}}{dt} = \frac{F}{m_0 g} - \frac{m}{m_0} \tag{3.20}$$

$$\frac{m}{m_0 g} \frac{d\dot{y}^{(1)}}{dt} = - \frac{D}{D_0} \tag{3.21}$$

(3.20) is the zeroth approximation without drag, and (3.21) gives the drag correction. Integrating (3.21), assuming $\dot{y}_0 = 0$,

$$\dot{y}^{(1)} = -g \int_0 \left(\frac{D}{D_0}\right)\left(\frac{m_0}{m}\right) dt \tag{3.22}$$

But (D/D_0) is to be computed by using the altitude of the zeroth approximation at time t and similarly $m_0/m \sim t$ relation is specified by the thrust variation chosen. Then after the zeroth approximation has been computed, the first approximation can be evaluated as a simple quadrature, numerically in general. Another form of (3.22) is, using (3.5)

$$\dot{y}^{(1)} = -g \int_0 \left(\frac{D}{D_0}\right) e^{\frac{\dot{y}^{(0)} + gt}{c}} dt \tag{3.23}$$

A second integration of (3.22) and (3.23) gives the correction of altitude due to drag.

As an example, let us use a "constant acceleration" rocket, so that

$$\dot{y}^{(0)} = ngt \quad , \quad y^{(0)} = \tfrac{1}{2} ngt^2$$

And let the density of atmosphere decrease exponentially, and the drag coefficient be constant, A the cross-sectional area

$$D = \frac{1}{2}\rho_0 e^{-\alpha y^{(0)}} \left[\dot{y}^{(0)}\right]^2 A C_D$$

Thus

$$\dot{y}^{(1)} = -\frac{\frac{1}{2}\rho_0 A C_D g}{D_0} \int_0 n^2 g^2 t^2 e^{-\frac{\alpha}{2} ngt^2 + \frac{(n+1)gt}{c}} dt \tag{3.24}$$

Or

$$\dot{y}^{(1)} = -\frac{\frac{1}{2}\rho_0 A C_D g}{D_0} \int_0 \frac{ngt}{\alpha}\left[-d\, e^{-\left\{\frac{\alpha}{2} ngt^2 - \frac{(n+1)gt}{c}\right\}} + \frac{(n+1)g}{c} e^{-\left\{\frac{\alpha}{2} ngt^2 - \frac{(n+1)gt}{c}\right\}} dt\right]$$

$$\dot{y}^{(1)} = -\frac{\frac{1}{2}\rho_0 A C_D g}{D_0}\left[-\frac{ngt}{\alpha} e^{-\left\{\frac{\alpha}{2} ngt^2 - \frac{(n+1)gt}{c}\right\}} + \frac{ng}{\alpha}\int_0 \left(1 + \frac{(n+1)gt}{c}\right) e^{-\left\{\frac{\alpha}{2} ngt^2 - \frac{(n+1)gt}{c}\right\}} dt\right]$$

But $\displaystyle\int_0 \left(1 + \frac{(n+1)gt}{c}\right) e^{-\left\{\frac{\alpha}{2} ngt^2 - \frac{(n+1)gt}{c}\right\}} dt$

$$= \frac{n+1}{n}\frac{1}{\alpha c}\int_0 \left[-d\, e^{-\left\{\frac{\alpha}{2} ngt^2 - \frac{(n+1)gt}{c}\right\}} + \frac{(n+1)g}{c} e^{-\left\{\frac{\alpha}{2} ngt^2 - \frac{(n+1)gt}{c}\right\}} dt\right]$$

$$+ \int_0 e^{-\left\{\frac{\alpha}{2} ngt^2 - \frac{(n+1)gt}{c}\right\}} dt$$

$$= \frac{n+1}{n}\frac{1}{\alpha c}\left[1 - e^{-\left\{\frac{\alpha}{2} ngt^2 - \frac{(n+1)gt}{c}\right\}}\right] + \left\{\frac{(n+1)^2}{n}\frac{g}{\alpha c^2} + 1\right\}\int_0 e^{-\left\{\frac{\alpha}{2} ngt^2 - \frac{(n+1)gt}{c}\right\}} dt$$

$$= \left(\frac{n+1}{n}\right)\frac{1}{\alpha c}\left[1 - e^{-\left\{\frac{\alpha}{2} ngt^2 - \frac{(n+1)gt}{c}\right\}}\right] + \left\{\frac{(n+1)^2}{n}\frac{g}{\alpha c^2} + 1\right\} e^{\frac{\alpha}{2} ng\left(\frac{n+1}{n}\right)^2 \frac{1}{c^2\alpha^2}} \int_0 e^{-\frac{\alpha}{2} ng\left\{t - \left(\frac{n+1}{n}\right)\frac{1}{c\alpha}\right\}^2} dt$$

Therefore

$$\dot{y}^{(1)} = -\frac{\frac{1}{2}\rho_0 A C_D g}{D_0}\,\frac{ng}{2}\Bigg\{ -t\,e^{-\left\{\frac{\alpha}{2}ngt^2 - \frac{(n+1)gt}{c}\right\}} + \left(\frac{n+1}{n}\right)\frac{1}{\alpha c}\left[1 - e^{-\left\{\frac{\alpha}{2}ngt^2 - \frac{(n+1)gt}{c}\right\}}\right]$$

$$+\left[\frac{(n+1)^2}{n}\frac{g}{\alpha c^2}+1\right]\frac{\sqrt{\pi}}{2}\sqrt{\frac{2}{\alpha ng}}\left[\operatorname{erf}\left\{\sqrt{\frac{\alpha ng}{2}}\left(t-\frac{n+1}{n}\frac{1}{\alpha c}\right)\right\}+\operatorname{erf}\left(\sqrt{\frac{\alpha ng}{2}}\,\frac{n+1}{n}\,\frac{1}{\alpha c}\right)\right]\Bigg\} \tag{3.25}$$

where erf is the error function defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-s^2}\,ds$$

In the case considered, the correction can thus analytically be calculated.

Problem 3.3

Assume exponential decrease of density with altitude and constant drag coefficient, develope the equation for the first order drag correction for altitude of a large "constantly accelerated" rocket.

Answer

$$y^{(1)} = -\frac{\frac{1}{2}\rho_0 A C_D g}{D}\,\frac{1}{\alpha^2}\Bigg\{\left[e^{-\left\{\frac{\alpha}{2}ngt^2 - \frac{(n+1)gt}{c}\right\}} - 1\right]\left[\left(\frac{n+1}{c}\right)^2\frac{g}{\alpha n}+2\right]$$

$$+\frac{(n+1)tg}{c} + \sqrt{\frac{\pi\alpha ng}{2}}\,e^{\left(\frac{n+1}{c}\right)^2\frac{g}{2\alpha n}}\left[\operatorname{erf}\left\{\sqrt{\frac{\alpha ng}{2}}\left(t-\frac{n+1}{\alpha nc}\right)\right\}+\operatorname{erf}\left(\sqrt{\frac{\alpha ng}{2}}\,\frac{n+1}{\alpha nc}\right)\right]$$

$$\left[-\frac{(n+1)^2}{c\alpha n} + \left\{\left(\frac{n+1}{c}\right)^2\frac{g}{\alpha n}+1\right\}\left(t-\frac{n+1}{\alpha nc}\right)\right]\Bigg\}$$

Problem 3.4

Take the following numerical values for different quantities to calculate the velocity and the altitude at burn-out:

$$M_0 g = 112{,}000 \text{ lbs.}$$

$$\zeta = 0.803$$

$$c = 11{,}000 \text{ ft./sec.}$$

$$n = 6.2$$

$$\text{Diameter} = 8.41 \text{ ft.}$$

$$C_D = 0.574$$

$$\rho_0 = 0.002378$$

$$\frac{1}{\alpha} = 22{,}000 \text{ ft.}$$

$$g = 32.2 \text{ ft./sec.}^2$$

Answer

$$\dot{y}_1 \cong \dot{y}_1^{(0)} - \dot{y}_1^{(1)} = 15400 - 1{,}086 = 14{,}314 \text{ ft/sec.}$$

$$y_1 \cong y_1^{(0)} - y_1^{(1)} = 594{,}000 - 57{,}300 = 536{,}700 \text{ ft.}$$

Performance Parameters

It is evident from the discussion in the previous section that the effect of drag is always to reduce the velocity and altitude. Therefore we have two conflicting tendencies: In order to minimize the influence of gravity, the burning-time must be short. On the other hand, short burning-time means very high velocity at the lower altitudes where the air density is high and therefore high drag. Hence there must be an optimum acceleration for any given case.

Let us assume constant thrust and effective exhaust velocity. Then the equation of motion can be written as

$$\frac{d^2y}{dt^2} = \frac{F}{m} - g - \frac{1}{2}\rho\left(\frac{dy}{dt}\right)^2 A C_D \frac{1}{m} \tag{3.26}$$

Now if $n_0 g_0$ is the initial acceleration at t = 0 when $\frac{dy}{dt} = y = 0$

$$n_0 g_0 = \frac{F}{m_0} - g_0 \tag{3.27}$$

where g_0 is the gravity at sea-level. Now let us introduce the non-dimensional quantities as follows:

$$\eta = \frac{y}{R}$$
$$\tau = \frac{g_0 t}{c} \tag{3.28}$$

where R is the radius of earth. Then Eq. (3.26) becomes

$$\left(\frac{Rg_0}{c^2}\right)\frac{d^2\eta}{d\tau^2} = -\frac{1}{(1+\eta)^2} + \frac{1}{1-(n_0+1)\tau}\left\{(n_0+1) - \left(\frac{\frac{1}{2}\rho_0 a_0^2 A C_D^*}{m_0 g_0}\right)\left(\frac{C_D}{C_D^*}\right)\sigma\left(\frac{Rg_0}{c^2}\right)\left(\frac{Rg_0}{a_0^2}\right)\left(\frac{d\eta}{d\tau}\right)^2\right\} \tag{3.29}$$

where a_0 is the velocity of sound at sea-level and C_D^* is the drag coefficient at M = 1 and $\sigma = \rho/\rho_0$. The ratio C_D/C_D^* is a function of Mach number M (neglecting the Reynold's number effect) and

$$M^2 = \left(\frac{Rg_0}{c^2}\right)\left(\frac{Rg_0}{a_0^2}\right)\left(\frac{T_0}{T}\right)\left(\frac{d\eta}{d\tau}\right)^2 \tag{3.30}$$

For a specific atmosphere, σ and T/T_0 are given functions of η and Rg_0/a_0^2 is a fixed constant.

The value of τ_1 at burn-out is given by

$$\tau_1 = \frac{\zeta}{(n_0 + 1)} \tag{3.31}$$

Let

$$\Lambda = \tfrac{1}{2}\rho_0 a_0^2 A C_D^* / m_0 g_0 \tag{3.32}$$

Λ is the same ~~putaubation~~ perturbation parameter used for drag correction. Therefore for vertical flight of a rocket with constant thrust and exhaust velocity, the parameters of the problem are

$$\zeta, \ \frac{c^2}{Rg_0}, \ n_0, \ \Lambda \text{ and } \frac{C_D}{C_D^*}$$

If the rocket body has geometrical similarity, then C_D/C_D^* is a fixed function of M and thus drops out of the list. If we have determined the optimum value of n_0^* for such a group of rockets, it must be a function of ζ, c^2/Rg_0 and Λ. If we are considering the same propellant, then n_0^* is a function of ζ and Λ. Previous calculations by F. J. Malina and A. M. O. Smith give $n_0^* \sim 1$ for $\zeta \sim 0.7$ and $\Lambda \sim 3$. Later calculations by H. R. Ivey, E. N. Bowen, Jr. and L. F. Oborny give $n_0^* \sim 2$ for denser rockets.

It is of practical importance to point out that if we fix the weight m_0 of the rocket and ζ, the value of Λ is dependent upon the propellant density. This is because low density propellant requires larger volume of propellant tanks and hence large A and Λ. If we have a particular

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Refs: Malina and Smith: "Analysis of Sounding Rockets" J.Aero.Sciences 5:199(1932)
Tsien and Malina: J. Aero. Sciences 6: 50 (1939)
Ivey, Bowen, Jr. and Oborny: NACA TN 1401 (1947)

initial weight and summit altitude in mind, we can set up a new parameter which is a measure of this combined property of c and density. This kind of propellant evaluation has been done. However it remains to be pointed out that if the rocket is very large, the drag influence or Λ influence can be neglected. Then the best propellant is the propellant with highest c.

Optimum Thrust Programming

Eq. (3.3) can be written as

$$m_0 e^{\dot{y}_0/c} = \int_0^{t_1} \frac{D}{c} e^{\frac{\dot{y}+gt}{c}} dt + m_1 e^{\frac{\dot{y}_1+gt_1}{c}}$$

Now if $\dot{y}_0$ is not zero, this means the rocket must be boosted to this velocity by an initial impulse. If we consider the rocket motor to have variable thrust, starting with an infinite thrust to boost the rocket to velocity $\dot{y}_0$, then $m_0 e^{\dot{y}_0/c}$ is really the gross initial mass including the boosting: Therefore if we wish to calculate this gross weight, we write

$$m_0 = \int_0^{t_1} \frac{D}{c} e^{\frac{\dot{y}+gt}{c}} dt + m_1 e^{\frac{\dot{y}_1+gt_1}{c}} \tag{3.33}$$

The problem of optimum thrust programming can then be formulated as follows: Given m_1, c, g and the drag function $D(y,\dot{y})$, what should be the function $y(t)$, such that m_0 is a minimum ? The auxiliary conditions are $y(0)=0$ and that y_1 and $\dot{y}_1$ at end of burning must be such as to reach the specified summit altitude Y. To reach the given summit altitude with the specified M, and D, y_1 and $\dot{y}_1$ are related, say

$$\dot{y}_1 = \phi(y_1) \tag{3.34}$$

where ϕ is a given function. For instance, at very high altitudes where the

D is negligible due to low air density

$$\dot{y}_1 \cong \sqrt{2g(Y-y_1)} \tag{3.35}$$

To find the conditions for the solution of this variational problem, let the required function be

$$y = y(t)$$

with $y(0) = 0$ (3.36)

Now let there be an artitrary function $\eta(t)$ such that

$$\eta(0) = 0 \tag{3.37}$$

but otherwise completely unspecified. Then the "neighboring" functions to $y(t)$ can be constructed as

$$\bar{y}(t) = y(t) + k(\varepsilon)\eta(t) \tag{3.38}$$

where k is a parameter but not a function of time. Because of (3.37) $\bar{y}$ satisfies the initial condition $\bar{y}(0) = 0$. The duration of powered flight, or the burning time, for the optimum solution is t_1. For the neighboring solution, the burning time is $t_1+\varepsilon$. k is thus a function of ε. For the optimum solution, k and ε both vanish. Therefore

$$\begin{aligned} k(0) &= 0 \\ k(\varepsilon) &\cong \varepsilon k'(0) \end{aligned} \tag{3.39}$$

By considering only terms up to first order in ε

$$\bar{y}_1 = \bar{y}(t_1+\varepsilon) = y(t_1) + \varepsilon\dot{y}(t_1) + k'(0)\varepsilon\eta(t_1) \tag{3.40}$$

$$\dot{\bar{y}}_1 = \left(\frac{d\bar{y}}{dt}\right)_{t=t_1+\varepsilon} = \dot{y}(t_1) + \varepsilon\ddot{y}(t_1) + k'(0)\varepsilon\dot{\eta}(t_1) \tag{3.41}$$

where $\dot{\eta} = \frac{d\eta}{dt}$. However $\bar{y}_1$ and $\dot{\bar{y}}_1$ must satisfy Eq. (3.34) so that the neighboring solutions will represent rockets reaching the specified summit altitude, the stated auxiliary condition. Therefore

$$\dot{\bar{y}}_1 = \phi(y_1) + \left(\frac{d\phi}{dy}\right)_{y_1}(\bar{y}_1 - y_1) + \text{------}$$

$$= \dot{y}_1 + \left(\frac{d\phi}{dy}\right)_{y_1}(\bar{y}_1 - y_1) + \text{-------}$$

By using (3.40) and (3.41), the above equation gives

$$\left[\left(\frac{d\phi}{dy}\right)_{y_1}\eta(t_1) - \dot{\eta}(t_1)\right]k'(0) = \ddot{y}(t_1) - \left(\frac{d\phi}{dy}\right)_{y_1}\dot{y}(t_1) \qquad (3.42)$$

This equation then determines the value of $k'(0)$. With the $k'(0)$ so determined, the neighboring solutions will definitely satisfy all auxiliary conditions.

With $\eta(t)$ specified, the total initial mass m_0 will be dependent upon ε . Let

$$F(y, \dot{y}, t) = D(y, \dot{y})\, e^{\frac{\dot{y}+gt}{c}}$$

Then by substituting (3.38) and (3.41) into (3.33), we have

$$m_0(\varepsilon) = \frac{1}{c}\int_0^{t_1+\varepsilon} F(y+k\eta, \dot{y}+k\dot{\eta}, t)\,dt + m_1 e^{\frac{\dot{y}_1+\varepsilon\ddot{y}_1+k\dot{\eta}(t_1)+gt_1+g\varepsilon}{c}}$$

The condition for $y(t)$ to correspond to the optimum solution can now be stated simply as

$$\left(\frac{\partial m_0}{\partial \varepsilon}\right)_{\varepsilon=0} = 0$$

Carrying out the required differentiation, we get

$$\frac{1}{c}k'(0)\int_0^{t_1}\eta\left[\frac{\partial F}{\partial y}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right)\right]dt+\frac{1}{c}k'(0)\eta(t_1)\left(\frac{\partial F}{\partial \dot{y}}\right)_{t_1}$$

$$+\frac{1}{c}F(y_1,\dot{y}_1,t_1)+\frac{m_1}{c}\left[\ddot{y}_1+g+k'(0)\dot{\eta}(t_1)\right]e^{\frac{\dot{y}_1+gt_1}{c}}=0$$

But $\eta(t)$ is arbitrary other than the condition $\eta(0)=0$. Therefore in order for the equation to be true,

$$\frac{\partial F}{\partial y}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right)=0 \tag{3.43}$$

and

$$k'(0)\eta(t_1)\left(\frac{\partial F}{\partial \dot{y}}\right)_{t_1}+F(y_1,\dot{y}_1,t_1)+m_1\left[\ddot{y}_1+g+k'(0)\dot{\eta}(t_1)\right]e^{\frac{\dot{y}_1+gt_1}{c}}=0 \tag{3.44}$$

(3.43) is the familiar Euler-Lagrange differential equation. (3.44) is the result of the auxiliary condition of the problem, sometimes called the condition of transversality.

By eliminating $k'(0)$ between (3.42) and (3.44), one has

$$0=\left[\ddot{y}_1-\left(\frac{d\phi}{dy}\right)_{y_1}\dot{y}_1\right]\eta(t_1)\left(\frac{\partial F}{\partial \dot{y}}\right)_{t_1}+\left[\left(\frac{d\phi}{dy}\right)_{y_1}\eta(t_1)-\dot{\eta}(t_1)\right]F(y_1,\dot{y}_1,t_1)$$

$$+m_1\left[\left\{\left(\frac{d\phi}{dy}\right)_{y_1}\eta(t_1)-\dot{\eta}(t_1)\right\}(\ddot{y}_1+g)+\dot{\eta}(t_1)\left\{\ddot{y}_1-\left(\frac{d\phi}{dy}\right)_{y_1}\dot{y}_1\right\}\right]e^{\frac{\dot{y}_1+gt_1}{c}}$$

But $\eta(t)$ is arbitrary. Therefore, in order for the above equation to be true, the sums of quantities which multiply into $\eta(t_1)$ and $\dot{\eta}(t_1)$ must be zero separately. Therefore

$$\left\{\ddot{y}_1-\left(\frac{d\phi}{dy}\right)_1\dot{y}_1\right\}\left(\frac{\partial F}{\partial \dot{y}}\right)_1+\left(\frac{d\phi}{dy}\right)_1F(y_1,\dot{y}_1,t_1)+m_1\left(\frac{d\phi}{dy}\right)_1(\ddot{y}_1+g)e^{\frac{\dot{y}_1+gt_1}{c}}=0 \tag{3.45}$$

and

$$F(y_1, \dot{y}_1, t_1) + m_1\left[g + \left(\frac{d\phi}{dy}\right)_1 \dot{y}_1\right] e^{\frac{\dot{y}_1 + gt_1}{c}} = 0 \tag{3.46}$$

Equations (3.43), (3.45) and (3.46) are now the complete answer to the variational problem. That is, the thrust programming must be such that Eq. (3.43) is satisfied at every instant of the powered flight and in addition at the end of the powered flight, the condition given by (3.45) and (3.46) must be satisfied.

These conditions can be reduced to simpler forms if the relation for F and D is reintroduced. Then (3.43) becomes

$$\frac{\partial D}{\partial y} = \frac{\partial^2 D}{\partial y \partial \dot{y}} \dot{y} + \frac{\partial^2 D}{\partial \dot{y}^2} \ddot{y} + \frac{1}{c}\left\{\frac{\partial D}{\partial y} \dot{y} + \frac{\partial D}{\partial \dot{y}}(2\ddot{y} + g) + \frac{D}{c}(\ddot{y} + g)\right\} \tag{3.47}$$

when the drag is specified as a function of y and $\dot{y}$, (3.47) gives the differential equation for the trajectory $y(t)$.

(3.45) and (3.46) now become

$$\left\{\ddot{y}_1 - \left(\frac{d\phi}{dy}\right)_1 \dot{y}_1\right\}\left\{\left(\frac{\partial D}{\partial \dot{y}}\right)_1 + \frac{D(y_1, \dot{y}_1)}{c}\right\} + \left(\frac{d\phi}{dy}\right)_1 D(y_1, \dot{y}_1) + m_1\left(\frac{d\phi}{dy}\right)_1 (\ddot{y}_1 + g) = 0 \tag{3.48}$$

and

$$D(y_1, \dot{y}_1) + m_1\left[\left(\frac{d\phi}{dy}\right)_1 \dot{y}_1 + g\right] = 0 \tag{3.49}$$

(3.49) is however automatically satisfied if (3.34) specifies the trajectory during coasting. The differential equation during coasting can be deduced from (3.17) by putting $F = 0$.

$$m_1 \frac{d\dot{y}_1}{dt} = -m_1 g - D$$

Or

$$m_1 \dot{y}_1 \left(\frac{d\phi}{dy}\right)_1 = -m_1 g - D(y_1, \dot{y}_1)$$

This is exactly (3.49). By eliminating $\left(\frac{d\phi}{dy}\right)_1$, between (3.48) and (3.49), the condition at the end of the powered flight is finally expressed as

$$\dot{y}_1 (m_1\ddot{y}_1 + D_1 + m_1 g)\left\{\left(\frac{\partial D}{\partial \dot{y}}\right)_1 + \frac{D_1}{c}\right\} = (m_1\ddot{y}_1 + D_1 + m_1 g)(D_1 + m_1 g)$$

But $m_1\ddot{y}_1 + D_1 + m_1 g$ is equal to the thrust just before burn-out, and is thus usually not zero. Hence the above equation is equivalent to

$$\dot{y}_1 \left\{\left(\frac{\partial D}{\partial \dot{y}}\right)_1 + \frac{D_1}{c}\right\} = D_1 + m_1 g \tag{3.50}$$

The problem of optimum thrust programming can now be discussed in more concrete terms: Since the aerodynamic drag D enters linearly and homogeneously into Eq. (3.47), that equation is actually a second order differential equation for $y(t)$, independent of the size of the rocket body. However, being a second order differential equation, with only one initial condition $y(0) = 0$, the initial velocity or velocity after boosting $\dot{y}_0$ is yet free and undetermined. It is determined, however, by the condition (3.50) at the end of the powered flight. In other words, for a given size of the rocket and for a given final mass m_1 , there is a corresponding optimum booster velocity $\dot{y}_0$ and subsequent optimum thrust programming for any specified summit altitude Y. In general then, the optimum solution always involves an impulsive start from rest. This is a characteristic of the problem.

G. Hamel* has studied the problem of optimum thrusting programming as early as 1927. However his solution is incomplete, for not giving the condition (3.50). More recent complete investigation** shows that the gain by optimum thrust programming is not large. However for very large rockets of long range, the pay load is only a verly small fraction of the gross weight. Then even a small gain in therms of the gross weight could give a substantial improvement in pay load ratio. The interesting characteristics of the optimum thrust solutions are 1) the considerable initial boosting to approximately 1000 fps. and 2) the subsequent smooth thrust variation with rapidly increasing thrust against altitude. Whether such ideal optimum solutions are feasible in engineering practice is a question yet to be answered.

Problem 3.5

Set up and derive the equations for optimum thrust programming by fixing Y, m_0 and the drag function $D(y, \dot{y})$.

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* G. Hamel, "Über eine mit dem Problem der Rakete zusammenhängende Aufgabe der Variationsrechnung". ZaMM 7:451 (1927)

** Tsien and Evans, J. Am. Rocket Soc. 21:99 (1951)

4. Coasting Rocket Trajectories

In the following series of lectures, we shall consider rocket trajectories for range over the surface of the earth. If maximum range with a given rocket is the aim, then in general we have two questions to ask: What is the best power trajectory? What is best coasting trajectory? We have seen in the last section that with relatively small penalty in loss of velocity due to gravity, and due to air drag, we can penetrate the dense layer of the atmosphere with a resulting high velocity. For long range rockets, it is certain that the optimum power trajectory will be relatively short and the purpose is to achieve high velocity at burn-out. Therefore the vertical powered trajectory must be close to the optimum powered trajectory even for long-range rockets. The actual optimum trajectory will be slightly curved by using the aerodynamic lift forces of the wings and the body, and the gravitational forces. We shall assume here that the velocity at burn-out for such curved trajectory can be approximated by the velocity for vertical trajectory calculated previously. The inclination of the trajectory at burn-out is so controlled as to give maximum range by coasting, neglecting the contribution to range during powered flight. This would be good approximation for long-range rockets.

The remaining question is then that optimum coasting trajectory for range with fixed initial velocity. In a scale of thousands of miles, the dense layer of air being only a few tens of miles thick is negligibly thin. An approximate picture would be to consider all aerodynamic forces to be available only at "zero" altitude. Under such circumstances, the best trajectory definitely include a high-soaring loop, with negligible drag. This is approximated by elliptic vaccum trajectory. We shall consider this part of coasting trajectory first.

Elliptic Trajectory

The effects of the rotation of the earth will be neglected first.

The flight is considered to have started at the surface of the earth with a velocity v_0 which is inclined to the surface by an angle ψ. Use polar coordinate and let $\theta = 0$ corresponds to the summit of the trajectory. $\theta = -\theta_1$ is the value of θ at the starting point of the flight. Due to symmetry of the trajectory, the range S_1 is

$$S_1 = 2R\theta_1 \tag{4.1}$$

where R is again the radius of the earth.

Let the mass of the vehicle be m, then the force of attraction from the center of the earth is, according to the Newtonian law of gravitation

$$\frac{km}{r^2}$$

where k is the constant. The potential energy at r, due to this force of attraction is equal to the negative of the work done in letting the mass "fall" from infinity to r. Thus

$$\text{Potential Energy} = +\int_{\infty}^{r} \frac{km}{r^2}(+dr) = km\left[\frac{-1}{r}\right]_{\infty}^{r} = -\frac{km}{r} \tag{4.2}$$

The kinetic energy is given by

$$\text{Kinetic Energy} = \frac{1}{2}m\left[\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2\right] \tag{4.3}$$

With no air drag, the sum of potential energy and kinetic energy must be a constant equal to the value at the starting point. Hence

$$\frac{1}{2}\left[\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2\right] - \frac{k}{r} = \frac{1}{2}v_0^2 - \frac{k}{R} \tag{4.4}$$

The linear momentum in the circumferential direction is $mr\frac{d\theta}{dt}$. Therefore the moment of momentum is $mr^2\frac{d\theta}{dt}$. Now since the central attractive force will not act on this quantity, the moment of momentum will be constant and equal to the value at the starting point. Hence

$$r^2\frac{d\theta}{dt} = Rv_0 \cos\psi \tag{4.5}$$

(4.4) and (4.5) are the two first integrals of the dynamic system and is useful here to derive results desired.

Now

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt}$$

Therefore with (4.5), we have

$$\frac{dr}{dt} = \frac{dr}{d\theta}\left(\frac{Rv_0\cos\psi}{r^2}\right) \tag{4.6}$$

By substituting (4.5) and (4.6) into (4.4), we get

$$\left(\frac{dr}{d\theta}\right)^2\frac{R^2v_0^2\cos^2\psi}{r^4} + \frac{R^2v_0^2\cos^2\psi}{r^2} - \frac{2k}{r} = v_0^2 - \frac{2k}{R} \tag{4.7}$$

Solving for $\left(\frac{dr}{d\theta}\right)$

$$\left(\frac{dr}{d\theta}\right)\frac{Rv_0\cos\psi}{r^2} = \sqrt{v_0^2 - \frac{2k}{R} - \frac{R^2v_0^2\cos^2\psi}{r^2} + \frac{2k}{r}} \tag{4.8}$$

Or

$$d\theta = \frac{-d\left\{\frac{Rv_0\cos\psi}{r} - \frac{k}{Rv_0\cos\psi}\right\}}{\sqrt{\left(v_0^2 - \frac{2k}{R} + \frac{k^2}{R^2v_0^2\cos^2\psi}\right) - \left\{\frac{Rv_0\cos\psi}{r} - \frac{k}{Rv_0\cos\psi}\right\}^2}} \tag{4.9}$$

This equation can be integrated as

$$\theta + \pi = \cos^{-1}\left[\frac{\dfrac{R v_0^2 \cos\psi}{r} - \dfrac{k}{R v_0 \cos\psi}}{\sqrt{v_0^2 - \dfrac{2k}{R} + \dfrac{k^2}{R^2 v_0^2 \cos^2\psi}}}\right]$$

where the integration constant π is incorporated. This equation can be rewritten as

$$-\cos\theta = \frac{\dfrac{R v_0^2 \cos\psi}{r} - \dfrac{k}{R v_0 \cos\psi}}{\sqrt{v_0^2 - \dfrac{2k}{R} + \dfrac{k^2}{R^2 v_0^2 \cos^2\psi}}} \qquad (4.10)$$

Now at the surface of the non-rotating earth, the gravitational constant is g_0. therefore

$$g_0 = \frac{k}{R^2} \qquad (4.11)$$

Therefore (4.10) becomes

$$-\cos\theta = \frac{\left(\dfrac{R}{r}\right)\cos\psi - \left(\dfrac{g_0 R}{v_0^2}\right)\dfrac{1}{\cos\psi}}{\sqrt{1 - 2\left(\dfrac{g_0 R}{v_0^2}\right) + \left(\dfrac{g_0 R}{v_0^2}\right)^2 \dfrac{1}{\cos^2\psi}}} \qquad (4.12)$$

Or

$$\frac{r}{R} = \frac{\left(\dfrac{v_0^2}{R g_0}\right)\cos^2\psi}{1 - \sqrt{\left(\dfrac{v_0^2}{g_0 R}\right)^2 \cos^2\psi\left(1 - 2\dfrac{g_0 R}{v_0^2}\right) + 1}\cdot\cos\theta} \qquad (4.13)$$

This equation shows that r is a maximum when $\theta = 0$ as specified. Therefore our choice of the integration constant in (4.10) is correct.

Let

$$\mathcal{E}^2 = \left(\frac{v_0^2}{g_0 R}\right)^2 \cos^2\psi \left(1 - 2\,\frac{g_0 R}{v_0^2}\right) + 1 \qquad (4.14)$$

Then (4.13) is

$$\frac{r}{R} = \left(\frac{v_0^2}{R g_0}\right) \frac{\cos^2\psi}{1 - \mathcal{E}\cos\theta}$$

But $\frac{r}{R} = 1$ when $\theta = \theta_0$, therefore

$$1 = \left(\frac{v_0^2}{R g_0}\right) \frac{\cos^2\psi}{1 - \mathcal{E}\cos\theta_0} \qquad (4.15)$$

Hence

$$\frac{r}{R} = \frac{1 - \mathcal{E}\cos\theta_0}{1 - \mathcal{E}\cos\theta} \qquad (4.16)$$

For a specified value of v_0 and ψ , (4.14) gives $\mathcal{E}$. Then (4.15) gives θ_0 . With θ_0 we can calculate the range S_1 by (4.1). The summit radius r_1 is given by

$$\frac{r_1}{R} = \left(\frac{v_0^2}{R g_0}\right) \frac{\cos^2\psi}{1 - \mathcal{E}} \qquad (4.17)$$

It is to be noted that the value of $\mathcal{E}$ is less than 1 for elliptic trajectory concerned here. If $\mathcal{E} > 1$, the trajectory is hyperbodic and the vehicle will not return to earth. Thus for our case, $v_0^2/Rg_0 < 2$.

To calculate the time of flight $t^{(1)}$, the symmetry property of the path gives

$$t^{(1)} = 2\int_R^{r_1} \frac{dt}{d\theta}\frac{d\theta}{dr}\,dr$$

By (4.5) and (4.8), this relation can be written as

$$t^{(1)} = 2\int_R^{r_1} \frac{r\,dr}{\sqrt{(v_0^2-2g_0R)r^2+2g_0R^2r-R^2v_0^2\cos^2\psi}}$$

$$= 2\sqrt{\frac{R}{g_0}}\int_1^{r_1/R} \frac{\left(\frac{r}{R}\right)d\left(\frac{r}{R}\right)}{\sqrt{\left(\frac{v_0^2}{g_0R}-2\right)\left(\frac{r}{R}\right)^2+2\left(\frac{r}{R}\right)-\left(\frac{v_0^2}{g_0R}\right)\cos^2\psi}}$$

The result of the integration is, using (4.17)

$$t^{(1)} = \frac{2\sqrt{\frac{R}{g_0}}}{\left(2-\frac{v_0^2}{Rg_0}\right)}\left[\sqrt{\frac{v_0^2}{Rg_0}}\sin\psi + \frac{1}{\sqrt{2-\frac{v_0^2}{Rg_0}}}\left\{\frac{\pi}{2}-\sin^{-1}\left(\frac{1-\frac{v_0^2}{Rg_0}}{\varepsilon}\right)\right\}\right] \tag{4.18}$$

Optimum Elliptic Trajectory on Non-rotating Earth

For a non-rotating earth, the equations derived in the previous section suffice. Let us consider first the optimum trajectory without earth rotation. The optimum trajectory is the one of minimum v_0, for a given range, or given θ_0. By eliminating $\cos^2\psi$ between (4.14) and (4.15),

$$1-\varepsilon^2 = \frac{v_0^2}{g_0R}\left(2-\frac{v_0^2}{g_0R}\right)\cos^2\psi = \left(2-\frac{v_0^2}{g_0R}\right)(1-\varepsilon\cos\theta_0)$$

Solving for v_0^2/g_0R, we have

$$\frac{v_0^2}{g_0R} = 2 - \frac{1-\varepsilon^2}{1-\varepsilon\cos\theta_0} \tag{4.19}$$

For a fixed θ_0, we can determine the value of ε which will give minimum v_0. Let this optimum value of ε be ε^*, then

$$\left[\frac{\partial}{\partial\varepsilon}\left(\frac{v_0^2}{g_0R}\right)\right]_{\varepsilon=\varepsilon^*} = 0$$

Or

$$-2\varepsilon^*(1-\varepsilon^*\cos\theta_0)+(1-\varepsilon^{*2})\cos\theta_0=0$$

Thus the equation for ε^* is

$$\cos\theta_0\cdot\varepsilon^{*2}-2\varepsilon^*+\cos\theta_0=0 \tag{4.20}$$

The appropriate ε^* is obtained, remembering that $\varepsilon<1$.

$$\varepsilon^*=\frac{1-\sin\theta_0}{\cos\theta_0}$$

The value of ε^*, the corresponding minimum v_0^* is given by

$$\frac{v_0^{*2}}{g_0R}=2\tan\theta_0\left(\frac{1-\sin\theta_0}{\cos\theta_0}\right) \tag{4.21}$$

The corresponding value of $\cos^2\psi^*$ is, from (4.15)

$$\cos^2\psi^*=\frac{\cos^2\theta_0}{2(1-\sin\theta_0)}$$

Or

$$\tan\psi^*=\frac{1-\sin\theta_0}{\cos\theta_0}=\varepsilon^* \tag{4.22}$$

Therefore ψ^* is always less than $\frac{\pi}{4}$.

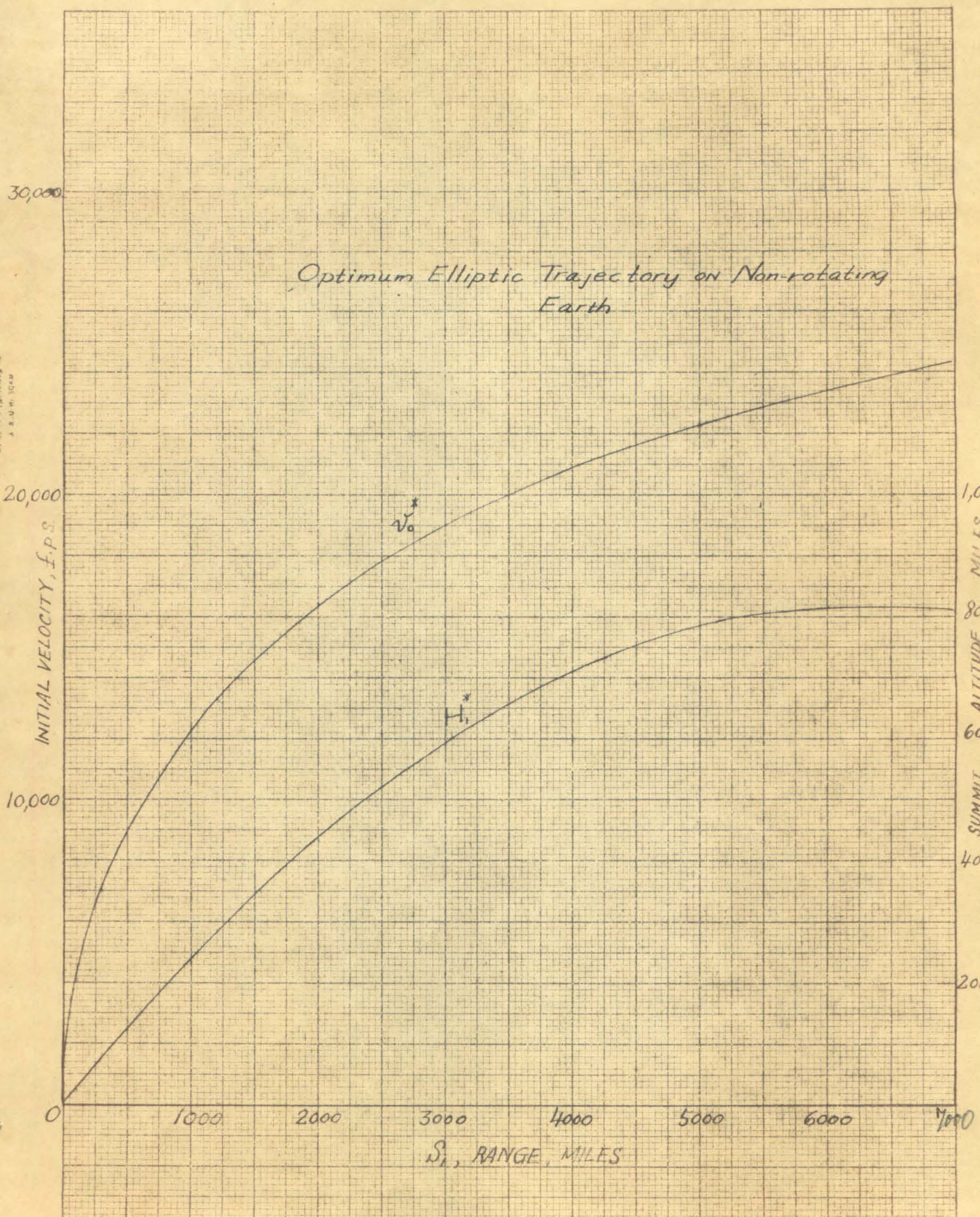
The summit altitude H_1^* is given by (4.17) and is

$$\frac{H_1^*}{R}=\left(\frac{v_0^{*2}}{Rg_0}\right)\frac{\cos^2\psi^*}{1-\varepsilon^*}-1=\frac{1-\sin\theta_0}{\cot\frac{\theta_0}{2}-1} \tag{4.23}$$

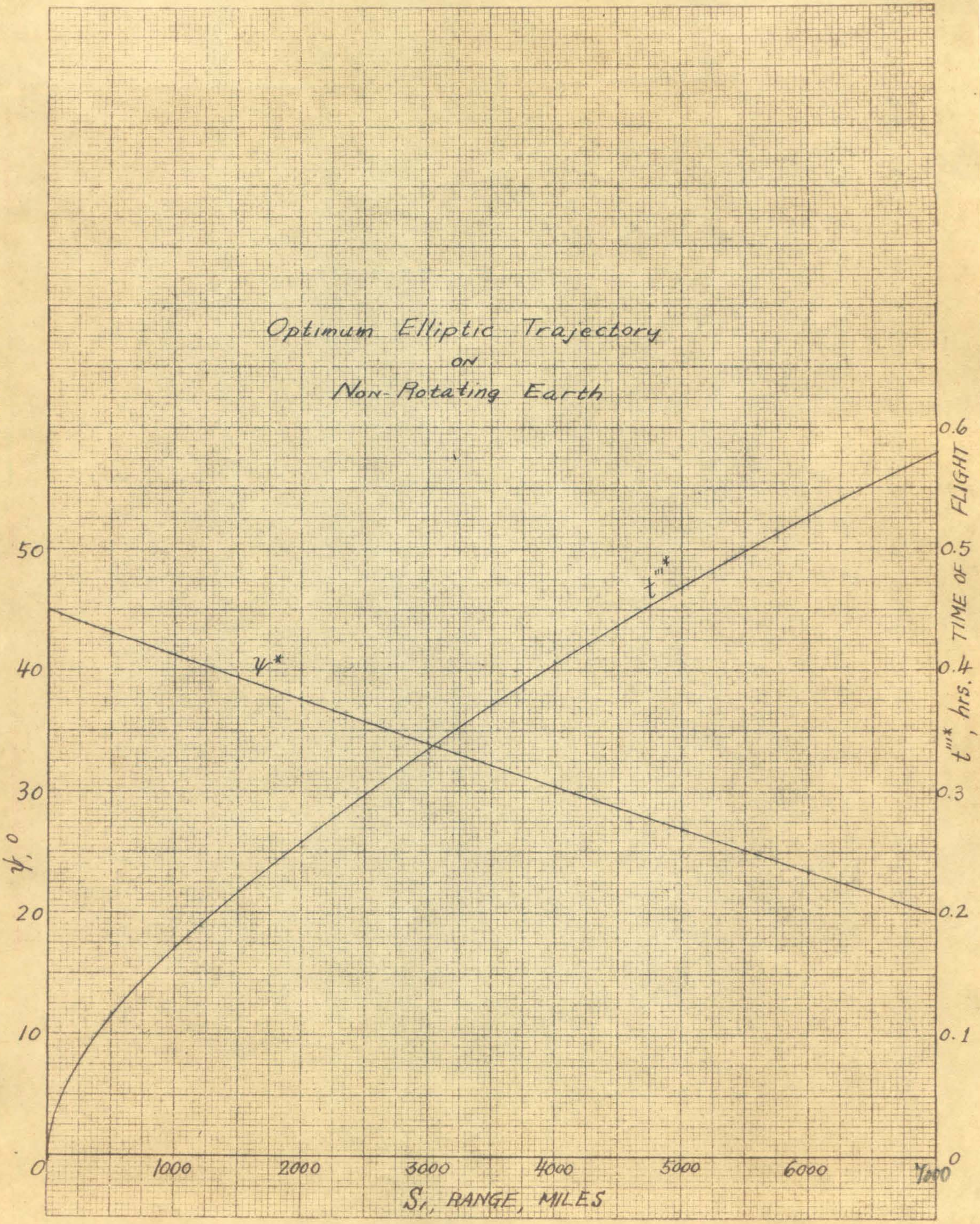
The time of flight of the optimum trajectory is calculated from (4.18). It is

$$t^{(1)*}=\sqrt{\frac{R}{g_0}}\,\frac{\cos^2\theta_0}{(1-\sin\theta_0)}\left[\frac{1-\sin\theta_0}{\cos\theta_0}\sqrt{\sin\theta_0}+\frac{1}{\sqrt{2}}\frac{\cos\theta_0}{\sqrt{1-\sin\theta_0}}\left\{\frac{\pi}{2}-\sin^{-1}\left(\frac{1-\sin\theta_0}{\cos\theta_0}\right)\right\}\right] \tag{4.24}$$

It is interesting to note that the previous equations all reduce to well-known relations for flat earth if the range or θ_0 is very small. For instance,



Optimum Elliptic Trajectory on Non-rotating Earth
30,000
20,000
10,000
0
INITIAL VELOCITY, f.p.s.
v_0^*
H_1^*
1,000
800
600
400
200
SUMMIT ALTITUDE MILES
1000
2000
3000
4000
5000
6000
7000
S_1, RANGE, MILES



Optimum Elliptic Trajectory
on
Non-Rotating Earth
ψ*
t'''*
ψ, °
t'''*, hrs. TIME OF FLIGHT
S₁, RANGE, MILES
0
1000
2000
3000
4000
5000
6000
7000
10
20
30
40
50
0.1
0.2
0.3
0.4
0.5
0.6

for small θ_0, (4.21) becomes

$$\left.\begin{aligned} v^{*2} &\cong g_0 S_1 \, , \\ \text{Then} \quad \psi^* &\cong \frac{\pi}{4} \, . \\ \text{And} \quad H_1 &\cong S_1/4 \end{aligned}\right\} \tag{4.25}$$

Problem 4.1

For a non-rotating earth,

$$g_0 = 32.2577 \text{ ft./sec.}^2$$
$$R = 3956.5 \text{ miles} = 20.88 \times 10^6 \text{ ft.}$$

Calculate the v_0^*, ψ^*, H_1^* and $t^{(1)*}$ for a range of 1600 miles and for $\theta_0 = \frac{\pi}{4}$.

Effect of Earth Rotation

Since we have assumed that the elliptic trajectory is high above the dense layer of atmosphere, the rotation of the earth has no effect on the motion of the vehicle. The calculation can thus be done most simply in non-rotating coordinates. This is the procedure adopted in the previous section. The effect of earth rotation is felt, however, both at the starting point of the trajectory and at the end point of the trajectory. At the starting point, the velocity to be supplied by the powered flight is not exactly the vector of magnitude v_0 inclined at angle ψ in the plane of the trajectory. The velocity to be supplied by the powered flight is the relative velocity to the rotating earth. The difference of the "absolute velocity" specified in the previous section and the relative velocity is the circumferential speed of rotation. This circumferential speed is $\Omega R \cos\phi$ where Ω is the angular velocity of earth rotation and ϕ is the angle of latitude, zero at equator, 90° at pole.

At the end point, the effect of earth rotation manifest itself as the shift in the effective range. If $t^{(1)}$ is the time of flight, then during $t^{(1)}$, the longitude has "rotated" by $\Omega t^{(1)}$. This may increase the range or decrease in range according to end point being west of the starting point or east of it. The same sense of change occurs in the relative velocity to be supplied by the powered flight. Therefore now the optimum trajectory will be different than the one determined in the previous section. Furthermore, the optimum trajectory will depend upon the orientation with respect to the axis of rotation of earth. The problem is thus complicated and perhaps the optimum trajectory can be only conveniently determined by graphical method.

Problem 4.2

Calculate the required relative initial velocity and the effective range for the 1600 mile absolute trajectory computed in Problem 4.1. Consider two cases only: Motion in equatorial plane, toward east and toward west.

Coasting Flight Around the Earth with Lift

Trajectories other than the elliptic trajectory will be carried out with the lift from the wing and the body. Such flight has to be within the atmosphere which rotates with the earth. Then the most convenient coordinate system will be that fixed with respect to the rotating earth. The equations of motion of a particle in such a coordinate system is complicated by the so-called Corioles forces. The following calculations are however only approximate and have the main purpose of comparing the ranges obtainable from different type of trajectories. We shall therefore simplify the analysis by assuming the earth to be non-rotating. This assumption is used for all following calculations.

Coasting flight around the earth with lift is considered to be carried out at a constant radius equal to R. The lift produced by the wing must counterbalance the resultant of the gravitational attraction and the centrifugal force. Actually the altitude of flight is not constant but so varied to enable the L/D ratio to be always at the maximum value. But since the atmosphere is relatively thin, this altitude variation is negligible in comparison with the range.

If m is the mass, the lift is

$$m\left(g_0 - \frac{u^2}{R}\right)$$

where u is the velocity, parallel to earth surface. Let the average L/D ratio during the flight be λ. Then the drag due to lift is

$$\frac{m}{\lambda}\left(g_0 - \frac{u^2}{R}\right)$$

The equation of motion is then

$$-m\frac{du}{dt} = -\frac{m}{2}\frac{du^2}{dx} = m\left(g_0 - \frac{u^2}{R}\right) \tag{4.26}$$

where x is distance along the earth. This equation can be written as

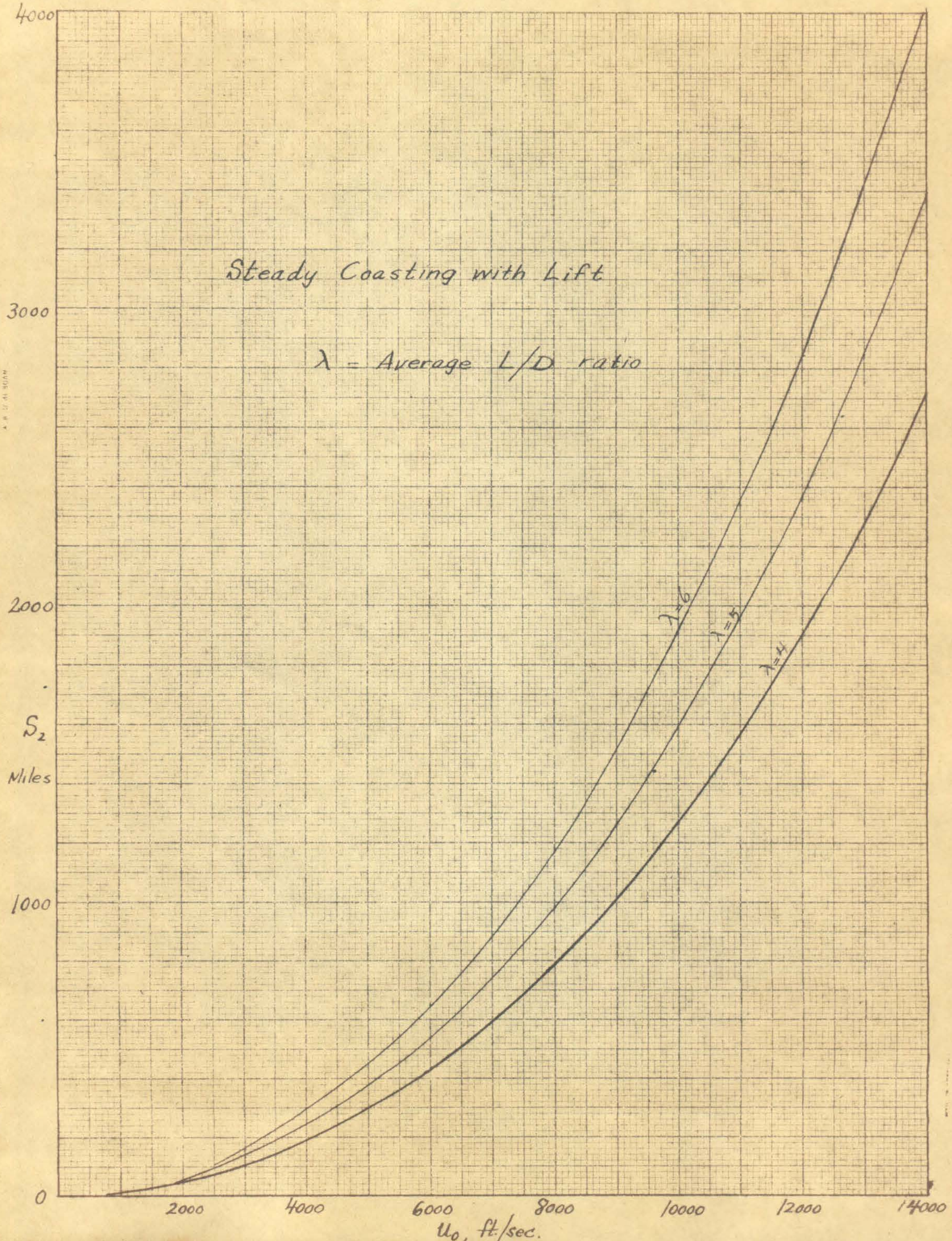
$$e^{\frac{2}{\lambda R}x}\frac{d}{dx}\left(e^{-\frac{2}{\lambda R}x}u^2\right) = -\frac{2g_0}{\lambda}$$

Thus

$$e^{-\frac{2}{\lambda R}x}u^2 = g_0 R e^{-\frac{2}{\lambda R}x} + \text{Constant} \tag{4.27}$$

The constant is determined by the initial condition that at $x=0$, $u=u_0$. Thus

$$u_0^2 = g_0 R + \text{Constant} \tag{4.28}$$



Steady Coasting with Lift
λ = Average L/D ratio
λ=6
λ=5
λ=4
4000
3000
2000
1000
0
S2
Miles
2000
4000
6000
8000
10000
12000
14000
u0, ft./sec.

By eliminating the constant between (4.27) and (4.28), we have

$$u^2 = g_0R - e^{\frac{2}{\lambda R}x}(g_0R - u_0^2) \qquad (4.29)$$

The range S_2 is determined by the end condtion: At $x = S_2$, $u = 0$. Then (4.29) gives

$$g_0R = e^{\frac{2}{\lambda R}S_2}(g_0R - u_0^2)$$

Hence by soliving for S_2,

$$S_2 = -\frac{\lambda R}{2}\log_e\left(1 - \frac{u_0^2}{g_0R}\right) \qquad (4.30)$$

The range is thus directly proportional to the average L/D ratio λ.

The maximum value of L/D is a function of Mach number and the Reynolds number. At high Mach numbers, the lift coefficient C_L at maximum L/D is quite small, say 0.08. So if the glide begins at a velocity of 12,000 ft./sec, and if the "wing loading" is 30 lbs./ft^2. Then the density ratio σ is determined by the relation

$$30 = 0.0011895 \times 12{,}000^2 \times 0.08\,\sigma$$

Thus

$$\sigma = \frac{30}{144 \times 1189 \times 0.08}$$

This corresponds to an altitude of approximately 140,000 ft. or 26 miles. If the wing chord is 10 ft., the Reynolds number based upon the wing chord would be approximately 2x10^6. Therefore we are still within the realm of gas dynamics. For a 5% thick wing at M=6.9, recent test shows* that L/D of 6 is possible. According to the hypersonic similarity law, some ratio will occur at higher Mach numbers with thinner sections. Therefore a vehicle of the dimensions considered, it value for λ should be between 4 and 6.

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* McLellan, C.H. "Exploratory Wind-Tunnel Investigation of Wings and Bodies at M = 6.9" J. Aero. Sc. 18: 641 (1951)

It is interesting to note that for small u_o, (4.30) can be written as

$$S_2 \cong \frac{\lambda}{2}\frac{u_o^2}{g_o}\left[1+\frac{1}{2}\left(\frac{u_o^2}{g_oR}\right)+\frac{1}{3}\left(\frac{u_o^2}{g_oR}\right)^2+\cdots\cdots\right] \qquad (4.31)$$

By comparing this equation with (4.25), we see that if $\lambda > 2$, the range obtained by gliding is larger than the range obtained by the elliptic trajectory high above the atmosphere, both with the same initial velocity. If we take only the initial term of (4.31), the result can be interpreted as the equality of the work done against drag to the initial kinetic energy. This is as follows: The lift at small u is roughly mg_o, the drag mg_o/λ. The total work done is thus $(mg_o/\lambda)S_2$. The kinetic energy is $\frac{1}{2}mu_o^2$. Hence the result. The additional terms in (4.31) represent the gain due to centrifugal force from the ~~con~~ cur vilinear motion of the vehicle around the earth.

To determine the time of flight, we observe that

$$dt = \frac{dt}{dx}dx = \frac{dx}{u}$$

Therefore the total time of flight $t^{(2)}$ is from (4.29)

$$t^{(2)} = \int_0^{S_2}\frac{dx}{u} = \frac{1}{\sqrt{g_oR}}\int_0^{S_2}\frac{dx}{\sqrt{1-\left(1-\frac{u_o^2}{g_oR}\right)e^{\frac{2}{\lambda R}x}}}$$

Therefore by letting

$$\xi = e^{\frac{2}{\lambda R}x}$$

the lower limit of integration in ξ is 1 and the upper limit is $\frac{1}{\left(1-\frac{u_o^2}{g_oR}\right)}$.

Hence

$$t^{(2)} = \frac{\lambda}{2}\sqrt{\frac{R}{g_o}}\int_0^{\frac{1}{\left(1-\frac{u_o^2}{g_oR}\right)}}\frac{d\xi}{\xi\sqrt{1-\left(1-\frac{u_o^2}{g_oR}\right)\xi}}$$

By carrying out the integration, we have

$$t^{(2)} = \frac{\lambda}{2}\sqrt{\frac{R}{g_0}}\,\log_e\left(\frac{1+\frac{u_0}{\sqrt{Rg_0}}}{1-\frac{u_0}{\sqrt{Rg_0}}}\right) \tag{4.32}$$

When the velocity is small, the above equation can be expanded,

$$t^{(2)} = \lambda\frac{u_0}{g_0}\left\{1+\frac{1}{3}\left(\frac{u_0^2}{Rg_0}\right)+\frac{1}{5}\left(\frac{u_0^2}{Rg_0}\right)^2+\frac{1}{7}\left(\frac{u_0^2}{Rg_0}\right)^3+\cdots\cdots\right\} \tag{4.33}$$

The loading term again be interpreted very simply: The drag is mg_0/λ . The deceleration is thus g_0/λ . The time to decelerate to zero velocity is thus $\lambda u_0/g_0$.

(4.30) and (4.31) show that the average speed of the coasting flight is independent of the L/D ratio and is equal to

$$\frac{S_2}{t^{(2)}} = \frac{u_0}{2}\left\{1+\frac{1}{6}\left(\frac{u_0^2}{Rg_0}\right)+\frac{7}{90}\left(\frac{u_0^2}{Rg_0}\right)^2+\cdots\cdots\cdots\right\} \tag{4.34}$$

The average speed is thus approximately half of the initial speed. This is considerably less than the average ground speed of the elliptic trajectory.

Pull-Outs during Coasting

To enter the glide from the elliptic flight path, we have first to turn the vehicle through an angle of approximately 40-45°. Similarly to join the different loops of a skip trajectory we also need such pull-ups. In this section, we shall calculate approximately such glide trajectories.

Since such a turning path is relatively short, the earth can be assumed to be flat and the gravitational field parallel. Let the radius of the circular path be r and the angle measured to the right of the verticle be θ. The velocity along the path is w. Then if m is the mass of the vehicle, the lift is equal to

$$m\left(g_0 \cos\theta + \frac{w^2}{r}\right)$$

Again let λ be the average L/D ratio during the flight. Then the drag is

$$\frac{m}{\lambda}\left(g_0 \cos\theta + \frac{w^2}{r}\right)$$

Therefore the equation of motion is

$$m\frac{dw}{dt} = -\frac{m}{\lambda}\left(g_0 \cos\theta + \frac{w^2}{r}\right) - mg_0 \sin\theta \tag{4.35}$$

But

$$w = r\frac{d\theta}{dt} = r\dot{\theta}$$

$$\frac{dw}{dt} = r\frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \frac{1}{2}\frac{d}{d\theta}(r\dot{\theta}^2)$$

$$\frac{w^2}{r} = r\dot{\theta}^2 \tag{4.36}$$

Therefore if we put

$$\eta = r\dot{\theta}^2 \tag{4.37}$$

(4.35) becomes

$$\frac{d}{d\theta}\left[e^{\frac{2}{\lambda}\theta}\eta\right] = -e^{\frac{2}{\lambda}\theta}\,2g_0\left(\frac{\cos\theta}{\lambda} + \sin\theta\right) \tag{4.38}$$

This equation can be easily integrated. The result is

$$\eta = Ae^{-\frac{2}{\lambda}\theta} - \frac{2g_0}{\left(1+\frac{4}{\lambda^2}\right)}\left\{\left(\frac{2}{\lambda^2}-1\right)\cos\theta + \frac{3}{\lambda}\sin\theta\right\} \tag{4.39}$$

where A is the integration constant.

To determine this constant, we utilize the initial condition that $w = w_0$ when $\theta = -\alpha$ the angle of entry into the circular part. Thus when $\theta = -\alpha$, $\eta = \frac{w_0^2}{r}$. Therefore

$$\frac{w_0^2}{r} = A e^{+\frac{2}{\lambda}\alpha} - \frac{2g_0}{(1+\frac{4}{\lambda^2})}\left\{\left(\frac{2}{\lambda^2}-1\right)\cos\alpha - \frac{3}{\lambda}\sin\alpha\right\} \tag{4.40}$$

By eliminating the constant A between (4.39) and (4.40), we have

$$\eta = e^{-\frac{2}{\lambda}(\theta+\alpha)}\left[\frac{w_0^2}{r} + \frac{2g_0}{(1+\frac{4}{\lambda^2})}\left\{\left(\frac{2}{\lambda^2}-1\right)\cos\alpha - \frac{3}{\lambda}\sin\alpha\right\}\right]$$
$$- \frac{2g_0}{(1+\frac{4}{\lambda^2})}\left\{\left(\frac{2}{\lambda^2}-1\right)\cos\theta + \frac{3}{\lambda}\sin\theta\right\} \tag{4.41}$$

This equation then gives the velocity along the flight path for any θ.

At the exit of the path, $\theta = \beta$ and $w = w_1$. Then (4.41) becomes

$$w_1^2 = e^{-\frac{2}{\lambda}(\alpha+\beta)} w_0^2 - \frac{2g_0 r}{(1+\frac{4}{\lambda^2})}\left\{\left(\frac{2}{\lambda^2}-1\right)\cos\beta + \frac{3}{\lambda}\sin\beta - e^{-\frac{2}{\lambda}(\alpha+\beta)}\left[\left(\frac{2}{\lambda^2}-1\right)\cos\alpha - \frac{3}{\lambda}\sin\alpha\right]\right\} \tag{4.42}$$

The horizontal distance covered is

$$S_3 = r\left(\sin\alpha + \sin\beta\right) \tag{4.43}$$

Now let us consider two special cases: To join the different loops of skip path, we need a pull-up path with $\alpha = \beta \cong \frac{\pi}{4}$. Let us then take $\alpha = \beta = \frac{\pi}{4}$. Futhermore, let the radius r of turning be such that

$$\frac{w_0^2}{r} = 6g_0$$

Then (4.42) becomes

$$\left(\frac{w_1}{w_0}\right)^2 = e^{-\frac{\pi}{\lambda}} - \frac{1}{3\sqrt{2}\left(1+\frac{4}{\lambda^2}\right)}\left\{\left(\frac{2}{\lambda^2}+\frac{3}{\lambda}-1\right) - e^{-\frac{\pi}{\lambda}}\left(\frac{2}{\lambda^2}-\frac{3}{\lambda}-1\right)\right\} \tag{4.44}$$

and

$$S_3 = \frac{w_0^2}{3\sqrt{2}\, g_0} = 0.2355\,\frac{w_0^2}{g_0} \tag{4.45}$$

If we take the average L/D ratio during the turning to be 6, then (4.44) gives

$$\left(\frac{w_1}{w_0}\right)^2 = 0.5052 \tag{4.46}$$

The second special case is the pull-out from the downward part of the elliptic trajectory to the horizontal glide. Here we take $\alpha = \frac{\pi}{4}$, $\beta = 0$, and

$$\frac{w_0^2}{r} = 3 g_0$$

Then (4.42) becomes

$$\left(\frac{w_1}{w_0}\right)^2 = e^{-\frac{\pi}{2\lambda}} - \frac{2}{3\left(1+\frac{4}{\lambda^2}\right)}\left\{\left(\frac{2}{\lambda^2}-1\right) - \frac{e^{-\frac{\pi}{2\lambda}}}{\sqrt{2}}\left(\frac{2}{\lambda^2}-\frac{3}{\lambda}-1\right)\right\} \tag{4.47}$$

and S_3 same as (4.45). In view of the fact that here the wing is also used to give lift during the horizontal glide, the L/D ratio during the pull-out must be less than the optimum. If we take $\lambda = 4$, then (4.47) gives

$$\left(\frac{w_1}{w_0}\right)^2 = 0.728 \qquad (\lambda = 4) \tag{4.48}$$

Comparison of the Skip Trajectory with Steady Glide Trajectory

The two types of trajectory which have to be compared are 1) the elliptic loop combined with the steady glide, 2) the skip trajectory* with multiple loops. These are the typical trajectory possible with winged rockets.

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* E. Sänger and J. Bredt, "A rocket Drive for Long Range Bombers" Translation CGD-32 (Buaer, Navy Department)

For the elliptic loop combined with steady glide, the range of the first loop with specified initial velocity v_0 is given by formula for the elliptic trajectory. Then turning path after the reentry into the dense atmosphere will begin with the velocity v_0, the same initial velocity, but the horizontal glide starts with a smaller velocity. If the pull-up is done at $3g_0$ and $\lambda = 4$, then the velocity reduction during the turn is specified by (4.48). The range during turn is given by (4.45). The range of the horizontal glide is then calculated by (4.30).

For the skip trajectory, the range of the first loop is approximately v_0^2/g_0. Actually the range is larger due to earth curvature and the reduction of g with altitude, as calculated in the previous section. We shall correct this later. The range covered in the first turning path is, according to (4.45)

$$0.2355 \frac{v_0^2}{g_0}$$

since the initial velocity v_0 is recovered when the vehicle re-enters the atmosphere. The total range covered in the first loop is then

$$1.2355 \frac{v_0^2}{g_0}$$

The second loop starts with a velocity v_1 which is smaller than the initial velocity v_0 due to the energy loss during pull-out. The ratio is given by (4.46) or

$$\left(\frac{v_1}{v_0}\right)^2 = 0.5052$$

Hence the range of second loop is only 50.52% of the first loop. By repeating this consideration, the range S of the complete trajectory is then

$$S = 1.2355 \frac{v_0^2}{g_0}\left\{1 + 0.5052 + (0.5052)^2 + (0.5052)^3 + \cdots\right\}$$

$$= \frac{v_0^2}{g_0} \frac{1.2355}{1 - 0.5052} = 2.50 \frac{v_0^2}{g_0}$$

But v_0^2/g_0 is considered as the range of the first loop or S_1 . Hence

$$S = 2.50\, S_1 \tag{4.49}$$

By this method of identifying v_0^2/g_0 with S_1 , we have corrected the error introduced at the beginning of this calculation. The multi-loop or skip trajectory is larger than the elliptic trajectory by 150%.

Actual numerical comparisons between the single loop with glide and the skip trajectory show that the difference is quite small. The second type is slightly favored in range. However due to the approximate nature of the analysis the true comparison can easily be otherwise. The important result is that a 150% increase in range is possible by using wings. The question of optimum glide trajectory for a pratical design remains as a unsolved variational problem.

Problem 4.3

Calculate the ranges of single loop and glide, and of skip trajectory for an initial velocity $v_0 = 14{,}000$ ft./sec. ($S_1 = 1{,}360$ miles)

Answer: a) Glide 3,200 miles b) Skip Trajectory 3,400 miles

Inefficiency of Horizontal Powered Flight

Calculations could be easily made to determine the performance of horizontal powered flight. It is found that this is highly inefficient for obtaining range. The reason for this lies in the characteristic of the rocket motor: The thrust is independent of the velocity of vehicle. When the rocket is outside the dense atmosphere and is thus not subject to drag loss, it is always advantageous to strive for high velocity during the application of rocket thrust. This will give more kinetic energy to the rocket and hence larger range. Mathematically, the change in velocity Δv is the same if the thrust is the same. But the change in kinetic energy is equal to $v \Delta v$. Hence the higher the velocity of flight the more efficiently the thrust is utilized.

Other Considerations Concerning Long Range Rockets

There is a prevailing feeling among aeronautical engineers that one of the most difficult problems in very high speed flight is the problem of high skin temperature at high speed. This can easily be seen when one remembers that for an "insulated" wall, the temperature is equal to the stagnation temperature. Of course actually insulated wall really means wall at temperature equilibrium. The stagnation temperature is calculated from the stagnation enthalpy which is equal to the enthalpy of the free stream plus the kinetic energy per unit mass of the free stream. At high Mach numbers, the stagnation enthalpy is thus very much larger than the free stream enthalpy. This would give normally very high skin temperature. However for very high speed flight at high altitude as discussed here, the situation is alleviated* by the following facts:

a) The density of the air is very low at high altitude, hence the stagnation enthalpy per unit volume is not large. The actual heat flux even at high stagnation temperature cannot be toc large. Thus the equilibrium state can be reached only after a long time interval, if the skin has a heat capacity.

b) There is heat loss due to radiation which is relatively more important at high altitudes than at low altitudes.

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* Cf. E. B. Klunker and H. R. Ivey: "An Analysis of Supersonic Aerodynamic Heating with Continuous Fluid Injection" NACA TR 990 (1950)

c) Due to low pressure at high altitudes, the dissociation of oxygen and nitrogen molecules into atoms is greatly promoted even at moderate temperatures. The dissociation is a endothermic reaction, especially for nitrogen. There is thus a tendency to reduce the equilibrium skin temperature by dissociation at high altitudes.

If humans are to be carried in the rocket, then the question of apparant g becomes important. The apparant g is a vector which is the difference of the acceleration vector and the gravitational vector. If the acceleration vector is equal to the gravitational vector, the apparant g disappears. The apparantly gravity free state appears thus when the body is in a state of free fall. This happens during the ellipitic trajectory. The main problem during zero g is the problem of orientation, since human orientation is all based upon the sense of gravity. This question is recently discussed by P. A. Campbell.*

Of more serious consequences is the high g state during the initial acceleration and during pull-outs. This is a problem studied for sometime in aviation medicine. The results were summerized by C. F. Lombard.** The problem for the designer of long-range man-carrying rocket is that of designing within the limits found by such experiments.

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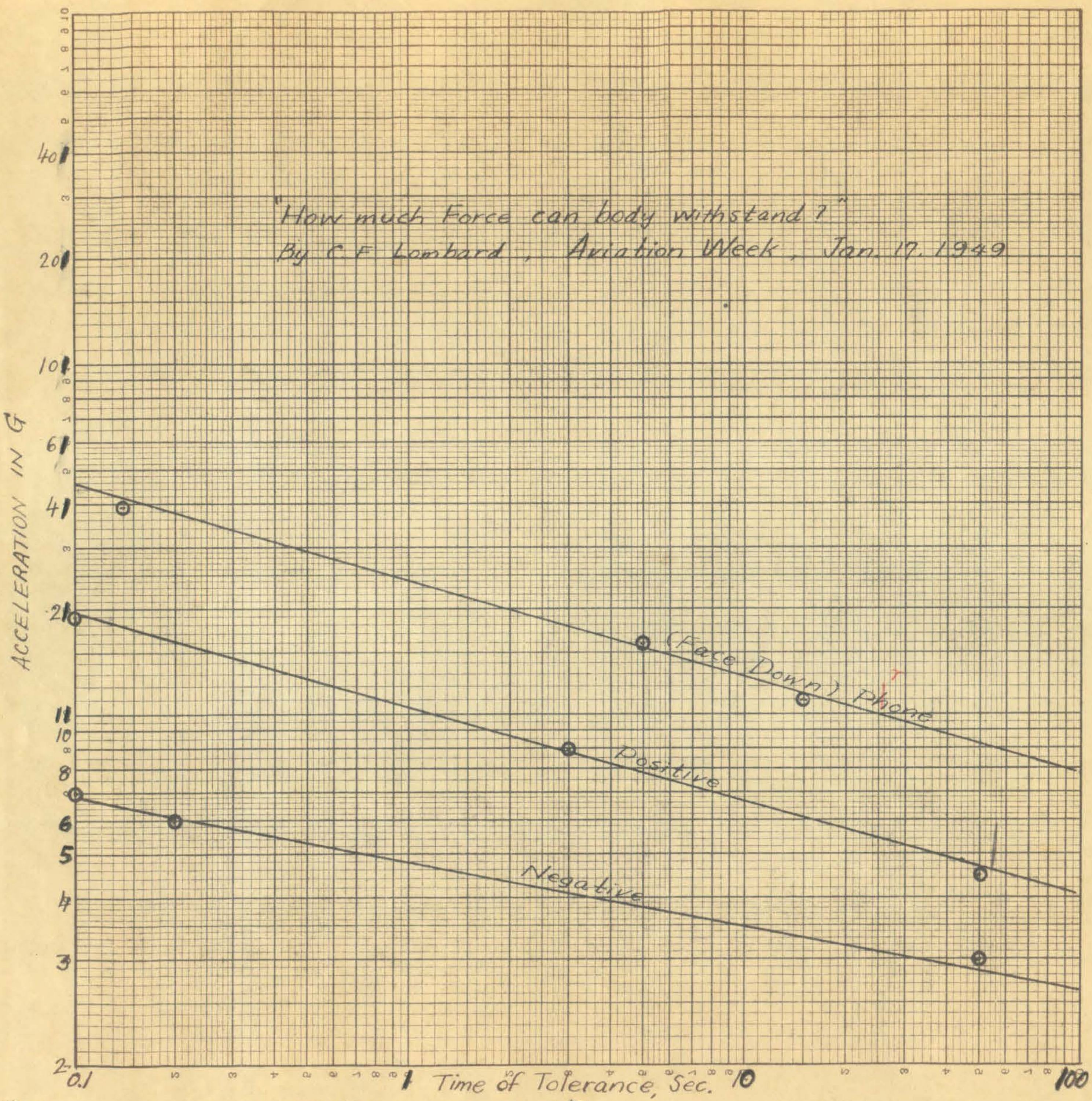
* See "Space Medicine", Edited by J. P. Marbarger, Univ. Ill. Press (1951) p.62

** C. F. Lombard, "How Much Force Can Body Withstand" Aviation Week. Jan. 17, 1949



"How much Force can body withstand?"
By C F Lombard, Aviation Week, Jan. 17. 1949
ACCELERATION IN G
Time of Tolerance, Sec.
(Face Down) Phone
Positive
Negative
0.1
1
10
100
2
3
4
5
6
8
10
11
21
41
61
101
201
401

5. Performance with Other Power Plants

In the previous chapters, we have studied the performance of vehicles powered by rocket. This choice of power plant is motivated by the desire to be able to carry out the analysis in the simplest manner. Of course, it is an open question as to whether rocket is the most suitable power plant for the purpose. Rocket is characterised by low installation weight but high consumption. Other power plants, such as ramjet, have a higher installation weight but lower consumption. It is the purpose of the present chapter to study the effects on performance when a power plant other than rocket is used.

Comparison of Power Plants

The various propulsive systems for aircraft generally differ in the type of fuels used, in the maintenance required, in the cost of operation, and in the complexity of the mechanisms involved. All these factors influence the particular choice of a propulsive system for a certain application. However, for a general discussion, we may choose as the primary criterion, the total initial weight of the system, i.e., the installed weight plus the weight of the propellants carried. Since the fuel weight increases with the duration of flight, the total initial weight of the propulsive system also increases with duration. Thus, if a very light propulsive system has a high fuel consumption, it will be only useful for short flight durations, as the fuel weight will soon cancel the advantage of the small installed weight of the system. It is then evident that the different propulsive system should be compared for various durations of flight.

The other parameters which effect the performance of the propulsive system are the velocity of flight and the altitude where the flight is carried out. Thus, for a preliminary classification of propulsive system, we should ask: Which system has the lowest total weight for the same thrust required at a chosen altitude, at a certain velocity and for a given duration? This question was studied by several authors*. The result of comparison is briefly the following: For very short durations of the order of a few minutes, the pure rocket motor gives the smallest total weight. For speeds over 400 mph, the ramjet is most suitable for durations of the order of 10 to 30 minutes, and the turbojet for longer durations. In general, the different power plants have their individual realms of optiumum operation in the three dimensional space of coordinates, duration, speed and altitude.

The above method of comparing the different power plants does not take into account the characteristics of the vehicle. For any given vehicle, flying a predetermined program, there is arelation between time, speed and altitude. In other words, the flight plane is a line through the space of time, speed and altitude. The true comparison of power plant is then much more complicated, a sort of average of the type of analysis indicated in the previous paragraph, weighted according to the particular application concerned.

As an example of this type of analysis. Let us consider the vertical flight of a vehicle. If c is now the average value of the "effective exhaust velocity", in ft./sec., i.e.,

$$c = \frac{3600\, g}{S}$$

where S the specific consumption of fuel or propellant in lbs. per hour per lb. of thrust, then according (3.6), the relation between the velocity $\dot{y}_1$, and time t_1 at burn-out to the mass ratio m_0/m_1, neglecting air drag, is

$$\log \frac{m_0}{m_1} = \frac{1}{c}\left(\dot{y}_1 + g t_1\right) = \left(\frac{\dot{y}_1 + g t_1}{3600\, g}\right) S \qquad (5.1)$$

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* For general method Cf: Th. von Kármán "Comparative Study of Jet Propulsion Systems as Applied to Missiles and Transonic Aircraft" J.P.L. Memorandum No. JPL-2 (1944)

Now let us consider a rocket and another vehicle propelled by a different power plant, both with the same gross weight, $m_0 g$. Let S^* be the rocket specific consumption ($\sim$10 to 18 lb./hr.$\sim$ lb.) and S be the specific consumption of the other power plant. Generally $S < S^*$. Therefore (5.1) shows that the final mass of the rocket m_1^*, must be smaller than the final mass m_1 of the other vehicle. This increase in the final mass means that the power plant is heavier. Now if m_2 is the structual mass plus the pay load mass, considered fixed, and m_3 the mass of the installed power plant, excluding fuel,

$$\left.\begin{aligned} m_1^* &= m_2^* + m_3^* = m_2 + m_3^* \\ m_1 &= m_2 + m_3 \end{aligned}\right\} \tag{5.2}$$

where m_3^* is the rocket engine mass. Now if both vehicles are to reach the same burn-out speed $\dot{y}_1$ at the same time instant t_1, (5.1) gives

$$\log \frac{1 + \left(\frac{m_3^*}{m_2}\right)\left(\frac{m_3}{m_3^*}\right)}{1 + \left(\frac{m_3^*}{m_2}\right)} = \left(1 - \frac{S}{S^*}\right) \log\left(\frac{m_0}{m_1}\right)^* \tag{5.3}$$

where $(m_0/m_1)^*$ is the mass ratio of the rocket, or $\frac{1}{1-\zeta}$. (5.3) can be solved for m_3/m_3^*.

$$\frac{m_3}{m_3^*} = \frac{1 + \left(\frac{m_2}{m_3^*}\right)}{(1-\zeta)^{\left(1-\frac{S}{S^*}\right)}} - \left(\frac{m_2}{m_3^*}\right) \tag{5.4}$$

The equation gives the value of m_3/m_3^* or the engine weight ratio which will give equal performance. If the actual engine weight ratio is less than this, then the pay load can be increased. The rocket then is not as good as the non-rocket vehicle. If the actual engine weight ratio is larger than this, then the rocket is the better engine.

As an example, take

$$\zeta = 0.75$$
$$m_2/m_3^* = 1.7$$
$$S^* = 14$$

If S = 4, then

$$\frac{m_3}{m_3^*} = 5.58$$

Let us consider the vehicle as the V-2 rocket, then the above values corresponding to a thrust of 55,000 lbs. and a rocket engine weight of 2,500 lbs. The new allowable engine weight is thus 13,950 lbs. The burn-out velocity of V-2 is 5,000 ft. If it is possible to operate the ramjet engine up to this speed, then certainly the ramjet engine weight will be less than 14,000 lbs. for 55,000 lbs. thrust. Then it would be advantageous to use the ramjet engine instead of the rocket as the power plant. The performance of ramjet in vertical accelerated flight was analysed under simplifying assumption by L. H. Schindel* and J. V. Rowny.** The investigation definitely indicates the possibility of using the ramjet for such dynamic trajectory and the possibility of using it as the first stage of a multi-stage rocket.

Improvement of Rocket Performance - Nuclear Rocket

The result (5.4) of the previous section can be also used to study the feasibility of any proposed method of improving the performance of rocket. For instance, the exhaust velocity c of a given propellant can be generally increased by increasing the combustion pressure. However this means stronger walls of the motor will be necessary. The injection pressure will be also

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* L. H. Schindel: "Application of Ramjet to High Altitude Sounding Vehicle" Thesis MIT (Aeronautical Engineering) (1948)

** J. V. Rowny: "Application of Ramjet to Vertical Ascent" Thesis CIT (Aeronautical Engineering) (1949) higher. All these will tend to increase the weight of power plant. The question is then: Will the advantage of increased c be cancelled by the increased weight of the power plant? This question can be answered by (5.4). Since the improvement in exhaust velocity or the specific consumption by increasing the combustion pressure is generally not large, we can simplify (5.4) by only retaining first order terms. Let the fractional <u>increment</u> of S be $\Delta S/S^*$; the fractional <u>increment</u> of engine weight by $\Delta m_3/m_3^*$. Then the relation between these two quanties to give same overall performance is

$$\frac{\Delta m_3}{m_3^*} = \left[\left(1 + \frac{m_2}{m_3^*}\right) \log(1-\zeta)\right] \frac{\Delta S}{S^*} \qquad . \quad (5.5)$$

In terms of increment in c

$$\frac{\Delta m_3}{m_3^*} = \left[\left(1 + \frac{m_2}{m_3^*}\right) \log\left(\frac{1}{1-\zeta}\right)\right] \frac{\Delta c}{c^*} \qquad (5.6)$$

Therefore if $\zeta = 0.75$, $m_3/m_3^* = 1.7$, then for 1% improvement in exhaust velocity, the weight of the engine can be increased by 3.75% with the same performance. If the actual design increase in engine weight is higher than this, the proposal is not worthwhile.

It is of interest to note that as the propellant loading ratio ζ increases, the advantage of higher exhaust velocity is greater. This means that high performance rockets are worthwhile to be improved, while low performance rockets are not. x

It has been proposed* to use a nuclear reactor in the form of a bundle of tubes with porous walls to heat hydrogen to extremely high temperature.

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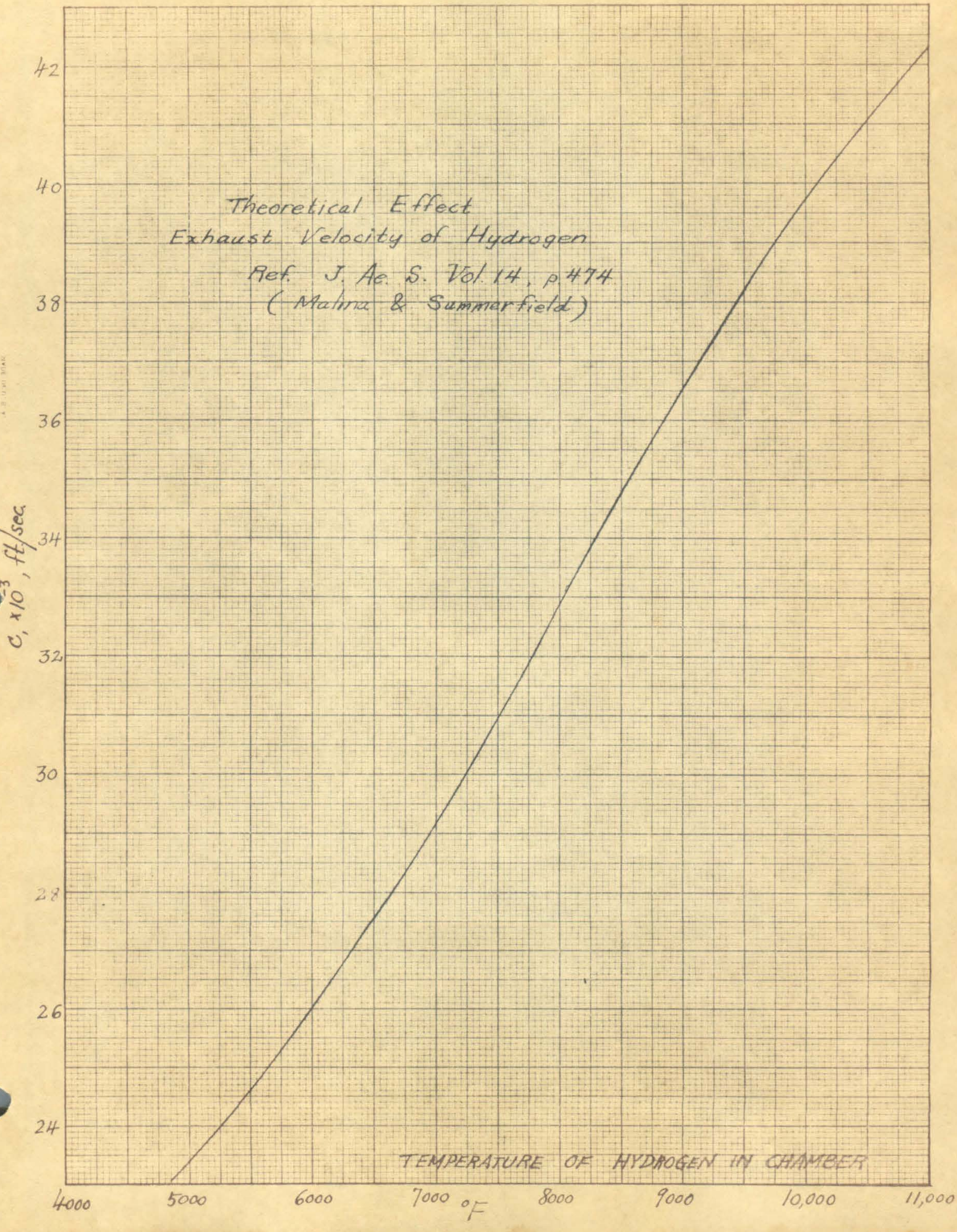
* "Science and Engineering of Nuclear Power" Addison-Wesley Press, Inc. Cambridge, Mass. Vol.II, P.124 (1949)
L. Green, "Gas Cooling of a Porous Heat Source" J. Appl. Mechanics (1952) (ASME Reprint 51-A-32)



Theoretical Effect
Exhaust Velocity of Hydrogen
Ref. J. Ae. S. Vol. 14, p 474
(Malina & Summerfield)
c, x10^-3, ft/sec
42
40
38
36
34
32
30
28
26
24
TEMPERATURE OF HYDROGEN IN CHAMBER
4000
5000
6000
7000
°F
8000
9000
10,000
11,000

The exhaust velocity would be quite large. But nuclear reactor is rather heavy. If radiation shield is required, the weight will be even higher. (5.4) can be again used to determine whether such a scheme is possible. For instance if heating up to 6000°F is possible, then according to Malina and Summerfield, c is 26,000 ft./sec., or S = 4.46 lb./hr.lb. Let $\zeta = 0.80$, $m_2/m_3^* = 1.7$, and $S^* = 10$. Then $m_3/m_3^* = 4.89$. Therefore if the use of nuclear reactor increases the weight of power plant system by more than 389%, nuclear rocket is not feasible. If less, the nuclear rocket has an advantage.

Inclined Thrust Axis for Lift

Let us consider the horizontal flight of a ramjet propelled aircraft. If V is velocity of flight, v the velocity of discharge and m the mass rate of air flow, then if we neglect the small fuel weight in the discharge, the thrust F is

$$F = m(v - V) \tag{5.7}$$

This assumes that the discharge nozzle has its axis in the direction of flight. If, however, the axis of the discharge nozzle is inclined at an angle θ so that the exhaust is discharged downward, the thrust F' is now

$$F' = m(v\cos\theta - V) \tag{5.8}$$

Furthermore now there is a lift force L' equal to

$$L' = m v \sin\theta$$

Admittedly, F' is less than F; but if the difference $F - F' = mv(1-\cos\theta)$ is considered as a drag, then the lift drag ratio is $\sin\theta/(1-\cos\theta)$ which is quite large for small θ. At high speed, the wing L/D ratio is only moderate, therefore there is a possibility of using an inclined nozzle to achieve a better overall L/D ratio and hence a saving in fuel. This idea was proposed by Benssuson ,* perhaps not the first time, and was analysed by F. Welch.**

Now let us consider an aircraft with body cross-sectional area A, wing area S. The drag coefficient of the body is C_{D_1}, the drag coefficient of the wing is C_{D_2}. The body is assumed to be non-lifting. The lift coefficient of the wing is C_L. Let $C_L/C_{D_1} = \lambda$, the L/D ratio of the wing. The weight is W. Then the equilibrium of drag and lift gives

$$\frac{1}{2}\rho V^2 A C_{D_1} + \frac{1}{2}\rho V^2 S C_{D_2} = m(v\cos\theta - V) \tag{5.9}$$

$$\frac{1}{2}\rho V^2 S C_L + m v \sin\theta = W \tag{5.10}$$

Now let us define

$$C_w = \frac{W}{\frac{1}{2}\rho V^2 A} \tag{5.11}$$

and

$$C_m = \frac{m}{\rho V A} \tag{5.12}$$

Then (5.9) and (5.10) can be written as

$$\frac{1}{2} C_{D_1} + \frac{1}{2} C_{D_2}\left(\frac{S}{A}\right) = C_m\left(\frac{v}{V}\cos\theta - 1\right) \tag{5.13}$$

$$\frac{1}{2} C_L\left(\frac{S}{A}\right) + C_m\left(\frac{v}{V}\right)\sin\theta = \frac{1}{2} C_w \tag{5.14}$$

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* French Patent

** F. Welch, "Analysis of an Inclined Thrust Axis as Applied to a Ramjet Propelled Aircraft". A.E. Thesis, C.I.T. (1950)

By eliminating (S/A) between (5.13) and (5.14), we get

$$\frac{1}{2} C_{D_1} + \frac{1}{2\lambda} C_w = C_m\left[\frac{v}{V}\left(\cos\theta + \frac{1}{\lambda}\sin\theta\right) - 1\right] \tag{5.15}$$

In order to have miminum fuel consumption, the air flow through the ramjet must be reduced. Thus the optimum condition of operation corresponds to minimum C_m. With C_{D_1} and C_w fixed by the given specification, we get minimum C_m by maximizing the quantity in the square bracket at the right of (5.15). Thus the optimum angle of the nozzle axis is

$$\tan\theta^* = \frac{1}{\lambda} \tag{5.16}$$

Thus

$$\frac{1}{2} C_{D_1} + \frac{1}{2\lambda} C_w = C_m^*\left[\frac{v}{V}\frac{\sqrt{\lambda^2+1}}{\lambda} - 1\right] \tag{5.17}$$

If $\theta = 0$, let the C_m be C_{m_0}. Then

$$\frac{1}{2} C_{D_1} + \frac{1}{2\lambda} C_w = C_{m_0}\left[\frac{v}{V} - 1\right] \tag{5.18}$$

Therefore

$$\frac{C_m^*}{C_{m_0}} = \frac{\frac{v}{V} - 1}{\frac{v}{V}\frac{\sqrt{\lambda^2+1}}{\lambda} - 1} \tag{5.19}$$

If $\frac{v}{V} = 1.3$, $\lambda = 6$, then $C_m^*/C_{m_0} = 0.944$. If $\lambda = 4$, then $C_m^*/C_{m_0} = 0.882$. Thus there is a fuel saving of 6% for $\lambda = 6$ and 12% for $\lambda = 4$. These savings are quite important if the fuel load is a large fraction of the gross weight. Then even a small percentage saving in fuel may mean a large percentage increase in pay load. The saving also increases if v/V or the thrust coefficient of the ramjet is smaller.

Needless to say, (5.19) really is applicable to any propulsion system. However for turbojets, $\frac{v}{V}$ is generally quite large. At subsonic or transonic speeds, λ is quite large. Then the savings in fuel consumption by inclining the discharge nozzle is negligible. For rockets, $\frac{v}{V} \to \infty$. Then the saving possible is again small. Furthermore, for rockets, the horizontal powered flight is very inefficient trajectory. Therefore although (5.19) is generally applicable, the only significant case is the ramjet powered aircraft.

6. Stability and Control - General Concepts

The performance analysis of a vehicle is only a first approximation to the engineering solution. It is a first approximation because we assume that all conditions are ideal, mainly without disturbing forces. In reality disturbing forces and incidental deviations from assumed circumstances always exist. The question is then whether the vehicle is stable in the sense that it will return to the ideal course, or how to control the vehicle so that it can be made to follow the desired course. These are the problems of stability and control. They are often more complicated than the problem of performance and require great skill in their solution. For instance, it has been said that while earlier inventors, Lilienthal, Mouillard and Chanute had carried out short soaring and gliding flights, it took the greater genius of the Wright brothers to really solve the control and stability problem. At present, the performance problem of long range flight at very high speed can be said to have been solved in its general outline. But the stability and control question is quite far from this satisfactory state of affairs.

Let us consider the simplest system - a system of one degree of freedom. Denote by y the deviation from the chosen course, t the time, and α the constant "spring constant", then the system without forcing function is described by

$$\frac{dy}{dt} + \alpha y = 0 \qquad (6.1)$$

The solution of this equation is

$$y = y^{o} e^{-\alpha t} \qquad (6.2)$$

where y^{o} is the deviation or "disturbed position" at time t = o, or simply the initial disturbance. Therefore it is evident from (6.2) that if

α is positive, the disturbance will be a decreasing function of time, and eventually the disturbance will disappear. The system is then said to be stable. If α is negative, the disturbance will be an increasing function of time, and eventually the disturbance will be so large as to be harmful to the proper operation of the system. The system is then said to be unstable. The singular case of $\alpha = 0$, is called neutral stability. In practice, only stable system is acceptable.

Now suppose the spring constant is a variable: The spring constant is in general a function of the condition of the system. For instance, the aerodynamic coefficients of an aircraft is a function of speed of the aircraft. If the speed of the aircraft is changing due to acceleration or deceleration, the aerodynamic coefficients will be changing. If we are considering the deviation from the chosen course, then the speed of aircraft can be translated into the time variable through the specified "normal course". Thus the aerodynamic coefficients or the spring constants are specified functions of time. In our simple system, the differential equation is

$$\frac{dy}{dt} + \alpha(t)\, y = 0 \tag{6.3}$$

The solution of this equation is

$$\log \frac{y}{y^{\circ}} = -\int_0^t \alpha(\xi)\, d\xi \tag{6.4}$$

where y° is again the initial disturbance. If α (t) is always positive, then log (y / y°) is always negative, then y / y° is always less than one. In fact, under such circumstances, y / y° is a decreasing function with respect to time. The motion is thus stable. If α (t) is always negative, the motion is unstable.

The interesting case is however when α (t) has both positive and negative values. Let α (t) to be first positive, then negative, but finally is positive again. This simulates the circumstance of a system going through a "unstable" region. Let t_1 corresponds to the first zero of α (t) and t_2 the second zero of α (t). Then it is easy to see that the minimum of x, x_{mi}, is given by

$$\log \frac{y_{mi}}{y^\circ} = -\int_0^{t_1} \alpha(\xi)\, d\xi \qquad (6.5)$$

The "maximum" of y, y_{ma} is given by

$$\log \frac{y_{ma}}{y^\circ} = -\int_0^{t_2} \alpha(\xi)\, d\xi \qquad (6.6)$$

The main question is whether the ratio y_{max} / y° is too large so that the system will not operate properly. It is entirely possible to have the integral in (6.6) positive even with a "unstable" region. Then the $\log(y_{ma}/y^\circ)$ will be negative due to the negative sign in front of the integral and $|y_{ma}| < |y^\circ|$. Then the system is satisfactory in spite of the fact that there is an unstable region.

If the integral in (6.6) is negative, then $|y_{ma}| > |y^\circ|$. We shall see that the ratio y_{ma}/y° can be controlled by the speed of going through the unstable region. Let $t = k\tau$, let α (t) is given by α (τ). Then (6.3) becomes

$$\frac{dy}{d\tau} + k\alpha(\tau) y = 0 \qquad (6.7)$$

Or

$$\log \frac{y_{ma}}{y^\circ} = -k\int_0^{\tau_2} \alpha(\xi)\, d\xi \qquad (6.8)$$

where τ_2 is the second zero of α (t). Therefore if k is large, then y_{ma}/y° can be very large if the integral is negative. k large means long time interval or going through the unstable region slowly. But if k is very small, the ratio $\mathcal{J}_{m\alpha}/\mathcal{J}$ is small even if the integral has a large negative value. Hence even if there is strong unstable region, the effect need not be catastrophic if the system can go through it rapidly.

An entirely different characteristic is exhibited by non-linear systems. For instance, if the spring constant α in our simple first degree system is a function of the disturbance y, then

$$\frac{dy}{dt} + \alpha(y)\,y = 0 \tag{6.9}$$

Let us put

$$\alpha(y)\,y = f(y) \tag{6.10}$$

Then the solution of (6.9) is specified by

$$t = -\int_{y^0}^{y} \frac{d\xi}{f(\xi)} \tag{6.11}$$

On the other hand, differentiation of (6.9) gives

$$\left.\begin{aligned} &\frac{d^2y}{dt^2} + \frac{df}{dy}\frac{dy}{dt} = 0 \\ &\frac{d^3y}{dt^3} + \frac{d^2f}{dy^2}\left(\frac{dy}{dt}\right)^2 + \frac{df}{dy}\frac{d^2y}{dt^2} = 0 \\ &- - - - - - - - - - - \end{aligned}\right\} \tag{6.12}$$

Thus if y_1 is a zero of the function f (t) and that if f (y) is regular at y_1, then all the derivatives of y with respect to t vanishes. Because in that case, all the derivatives of f with respect to y are finite at y_1. This means that y approaches y_1, asymptotically. In fact, if the initial disturbance y^0 is such that $y^0 > y_1$, and $f(y^0) > 0$, then the disturbance will become y_1 eventually. If $y^0 < y_1$, then $f(y^0) < 0$, the disturbance will become y_1 eventually also. This pattern is repeated with other zeros of f (y). The values of y for large time depends upon the initial values y^o but are always the zeros of the function $f(y)$. If y^o is a zero of $f(y)$ and this value of y^o will be maintained as time goes on. But if $\frac{df}{dy} < 0$ at this y^o, zero of $f(y)$, a slight change in the initial y^o produces drastic result: The final y will then be the next zeros of $f(y)$. If $\frac{df}{dy} > 0$, no such drastic result will occur. Such complicated pattern of behavior of the system is a characteristic of non-linear system.

The three examples illustrates the stability problem in three degrees of complexity and difficulty. If the disturbances are small and the characteristic time is long, such as airplane under wind disturbances, the problem is approximated by linear system with constant coefficients in the differential equation. If the time is short, such as a missile, the problem can be only solved by linear differential equations with coefficients that are functions of time. Finally if the disturbance is large, then the problem is non-linear. At present, only the first case is studied with a degree of completeness. This is the present stability and control theory of aircraft. A small beginning has been made for the second case. This is closely associated with the so-called ballistic disturbance theory in that we are concerned with the disturbance from the desired ballistic course. Much work however remains to be done before we can use the theory to solve practical problems. The non-linear problem so far seems to be much talked about and very little of it is solved.

7. Analysis of Systems with Constant Coefficients

In this chapter, we shall give an introduction to the present theory of stability and control based upon a system of linear ordinary differential equations with constant coefficients. The basic theory of the behavior of solutions of such systems is of course well-known and developed many years ago. The method used in the classical theory is however not practical when applied to engineering problems of very complicated nature. The main effort of modern servo-mechanism specialists is directed toward finding means of analysis which is applicable to complicated systems. The following introductory treatment follows closely a paper by W. Bollay.*

Laplace Transforms

The appropriate tool for linear equations of constant coefficients is the method of Laplace transform. If y (t) is function of t, defined for t $\geq$ o, the Laplace transform Y(p) is defined as

$$Y(p) = \int_0^\infty e^{-pt} f(t)\, dt \tag{7.1}$$

Any unit of a complicated system is characterized by an input y_i (t) and an output y_{ob} (t). They are related through a differential equation which characterizes the properties of that unit. Their Laplace transforms are $Y_i(p)$ and Y_{ob} (p):

$$Y_i(p) = \int_0^\infty e^{-pt} y_i(t)\, dt \tag{7.2}$$

$$Y_{ob}(p) = \int_0^\infty e^{-pt} y_{ob}(t)\, dt \tag{7.3}$$

We shall separate the output into two parts: The part y_o (t) is the output due to the input y_i (t). The part y_b (t) is due to the initial conditions of the system. In the language of classical theory, y_o(t) is the

* W. Bollay: "Aerodynamic Stability and Automatic Control" J. Ae. S. Vol. 18, pp. 569-617 (1951)

particular integral due to $y_i(t)$; $y_b(t)$ is the complementary function. Thus

$$y_{ob}(t) = y_o(t) + y_b(t) \tag{7.4}$$

and

$$Y_{ob}(p) = Y_o(p) + Y_b(p) \tag{7.5}$$

where

$$Y_o(p) = \int_0^\infty e^{-pt} y_o(t)\,dt \tag{7.6}$$

$$Y_b(p) = \int_0^\infty e^{-pt} y_b(t)\,dt \tag{7.7}$$

We now introduce a special terminology: The "gain" of the system is defined as

$$Y_o(0)/Y_i(0) = \lim_{p\to 0} Y_o(p)/Y_i(p) = K \tag{7.8}$$

Then

$$Y_o(p)/Y_i(p) = KG(p) \tag{7.9}$$

where $G(p)$ is called the "transfer function", or

$$G(p) = \frac{1}{K}\,\frac{Y_o(p)}{Y_i(p)} \tag{7.10}$$

It is evident that $y_i(t)$ and $y_{ob}(t)$ are not necessarily of the same dimension. Thus $Y_o(p)$ and $Y_i(p)$ are not necessarily of the same dimension. Hence the gain K is dimensional quantity, while the transfer function is non-dimensional.

The basic units a complicated system is resolved are first-order or second-order systems. In our simple analysis, we shall therefore study particularly the characteristics of the first-order and second-order systems and the methods for combining the solutions of these simple systems. The purpose of the analysis will be two-fold: (a) to determine the range of parameters for which the overall system is stable and (b) to determine the accuracy of control -

i.e., to compute the overall output response for a given input.

Analysis of First-Order Systems

Consider a cantilever spring which has a dashpot at its end. The spring stiffness is k, the damping constant of the dashpot is c. The other end of the cantilever spring is moved upward, its position being given by $y_i(t)$. The dashpot end lags behind the input due to the damping; it experiences the displacement $y_{ob}(t)$. (This simple case approximates, for example, the motion of a recording pen having viscous friction). Neglecting the mass-acceleration forces, the forces acting on the dashpot are

Spring force - upward = $k(y_i - y_{ob})$

Damping force - upward = $-c\, dy_{ob}/dt$

Thus, the equation of motion is

$$c\frac{dy_{ob}}{dt} + k y_{ob} = k y_i \tag{7.11}$$

Or by letting

$$T_1 = c/k \tag{7.12}$$

we have

$$T_1\frac{dy_{ob}}{dt} + y_{ob} = y_i \tag{7.13}$$

T_1 has thus the dimension of time, and called the characteristic delay time. The initial condition is

$$y_{ob} = y_b(0) \quad at \quad t = 0-\varepsilon \tag{7.14}$$

Multiply (7.13) by e^{-pt} and integrate from o to ∞, we have

$$(T_1 p + 1) Y_{ob}(p) = Y_i(p) + T_1 y_b(0) \tag{7.15}$$

and thus

$$Y_{ob}(p) = \frac{1}{T_1 p + 1} Y_i(p) + \left(\frac{T_1}{T_1 p + 1}\right) y_b(0) \tag{7.16}$$

Therefore $Y_o(p)$, representing the Laplace transform of the output due to y_i, is

$$Y_o(p) = \frac{1}{T_1 p+1} Y_i(p) \tag{7.17}$$

and $Y_b(p)$, representing the Laplace transform of the output due to initial conditions, is

$$Y_b(p) = \frac{T_1}{T_1 p+1} y_b(0) \tag{7.18}$$

From (7.17), the gain is 1, i.e.,

$$K = 1 \tag{7.19}$$

$$G(p) = \frac{1}{T_1 p+1} \tag{7.20}$$

The block diagram is thus as follows:

$$\xrightarrow[Y_i(p)]{} \boxed{G(p) = \frac{1}{T_1 p+1}} \longrightarrow Y_o(p)$$

$$Y_o(p) = \frac{1}{T_1 p+1} Y_i(p)$$

The transfer function $G(p)$ can be represented by a constant factor by the poles and zeros in the complex $p = \lambda + i\omega$ plane. The simple transfer function $G(p)$ (7.20) has no zero, it has a simple pole at $-\frac{1}{T_1}$.

Special Case: $y_i(t) = h(t)$

For the special case of a unit step input in displacement, we have

$$y_i(t) = h(t)$$

$$Y_i(t) = \int_0^\infty e^{-pt}\,dt = \frac{1}{p}$$

The corresponding solutions for the first order equation is thus

$$Y_o(p) = \frac{1}{p(T_1 p+1)} \tag{7.21}$$

and
$$y_o(t) = (1 - e^{-t/T_1}) \quad (7.22)$$

The other part of the output is

$$Y_b(p) = \frac{T_1}{T_1 p + 1} y_b(o) \quad (7.23)$$

and
$$y_b(t) = y_o(o) e^{-t/T_1} \quad (7.24)$$

Therefore the complete solution is

$$y_{ob}(t) = (1 - e^{-t/T_1}) + y_b(o) e^{-t/T_1} \quad (7.25)$$

The term $y_b(t)$ represents a damped subsidence due to the initial condition $y_b(o)$. This term enters into the analysis of automatic control systems only when it is necessary to compute the response of a system that is not initially at rest and in equilibrium.

The term $y_o(t) = 1 - e^{-t/T_1}$ represents the output motion due to the step input $h(t)$. It is represented by an exponential approach to the input. It indicates that the output lags behind the input, that at a time $t = T_1$, the output has reached 63% of its asymptotic deflection. The <u>error signal</u>, or the difference between input and output is

$$y_e(t) = y_i(t) - y_o(t) = e^{-t/T_1} \quad (7.26)$$

We note that the error $y_e(t) \rightarrow 0$ as $t \rightarrow \infty$ and that the initial slope of the output is

$$\left(\frac{dy_o}{dt}\right)_{t=o} = \frac{1}{T_1}$$

<u>Special Case:</u> $\quad y_i(t) = y_{im} e^{i\omega t}$

The special case of a sinusoidal input with frequency ω is most conveniently represented by writing the input $y_i(t) = y_{im} e^{i\omega t}$. Then

$$Y_i(p) = \frac{y_{im}}{p - i\omega}$$

Then the corresponding solution of the first order system is

$$Y_b(p) = \frac{T_1}{T_1 p + 1} y_b(0)$$
$$y_b(t) = y_b(0)\, e^{-t/T_1}$$

and $$Y_o(p) = y_{im} \frac{1}{T_1 p + 1} \frac{1}{p - i\omega} \qquad (7.27)$$

To find $y_o(t)$, we first observe that

$$Y_o(p) = y_{im}\left[-\frac{1}{1+i\omega T_1}\frac{1}{p+\frac{1}{T_1}} + \frac{1}{1+i\omega T_1}\frac{1}{p - i\omega}\right]$$

Therefore

$$y_o(t) = \frac{-y_{im}}{1+i\omega T_1} e^{-t/T_1} + \frac{y_{im}}{1+i\omega T_1} e^{+i\omega t} \qquad (7.28)$$

The terms Y_b (p) and y_b(t) are thus exactly the same as the previous special case. The solution y_o(t) has as its first term e^{-t/T_1}, a damped subsidence. This term represents a transcient disturbance associated with the initial application of the force.

The second term of y_o (t) represents a steady sinusoidal oscillation at the same frequency ω as the input. The ratio

$$\frac{[y_o(t)]_{steady}}{y_i(t)} = \frac{1}{1+i\omega T_1} \qquad (7.29)$$

It will be noted that this ratio is the same as would be obtained by writing $i\omega$ instead of p in G (p), i.e.,

$$G(i\omega) = \frac{1}{1+i\omega T_1}$$

and thus for a sinusoidal input

$$[y_o(t)]_{steady} = G(i\omega)\cdot y_i(t)$$

This form of relation is generally true.

It is convenient to express the complex vector $G(i\omega)$ in terms of its magnitude M and its phase angle θ - that is

$$G(i\omega) = Me^{i\theta} \tag{7.30}$$

In the above example of a first order system, we have then

$$Me^{i\theta} = \frac{1}{1+i\omega T_1} = \frac{1-i\omega T_1}{1+\omega^2 T_1^2}$$

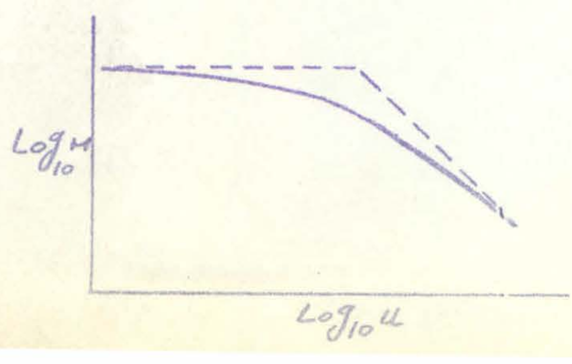
Or

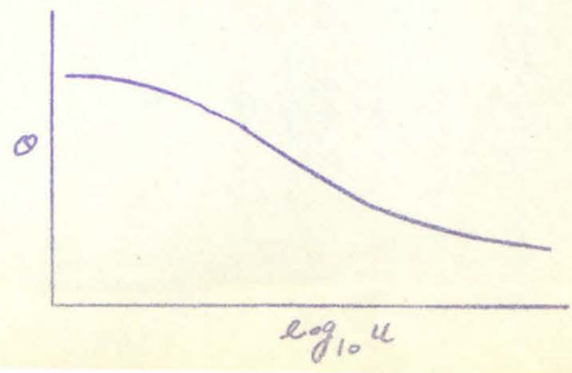
$$M = \frac{1}{\sqrt{1+\omega^2 T_1^2}} \tag{7.31}$$

and

$$\tan\theta = -\omega T_1 \tag{7.32}$$

The above relations indicate how well the output is able to follow a sinusoidal variation in the input. We note that at very low frequencies the steady output $y_o(t)$ follows the input $y_i(t)$ in magnitude and phase, at higher frequencies the ratio M becomes less than one and the output lags the input, and at very high frequencies $M \to 0$ and the phase angle lags by 90°. The variation of M and θ with the dimensionless frequency $u = \omega T_1$, can be on log-log paper. Then since as $u \to \infty$, $M \to \frac{1}{u}$, the graph of $\log_{10} M$ vs. $\log_{10} u$ has a slope of -1 for large u. For small values of u, $\log_{10} M$ starts with a slope of 0. Thus, the first-order system can be approximated by the two dotted straight lines on this diagram.

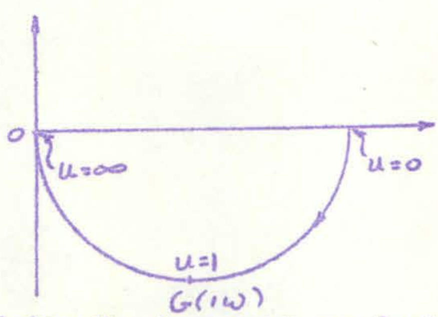


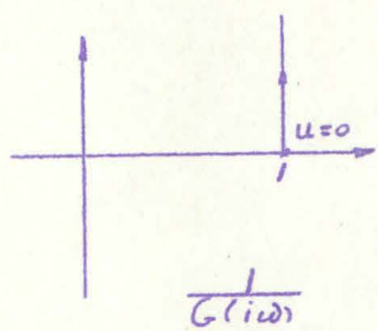


In the accoustic and electrical literature it is customary to plot $20 \log_{10} M$ in order to convert the amplitude units into decibels. A doubling in frequency is called an octave, and thus the region where our curve of $\log_{10} M$ vs $\log_{10} u$ has a slope of -1 is denoted by a slope of $-20 \log_{10} 2 = -6.02$ db. per octave.

It will be noted that on this logarithmic plot the approximate curve $\log_{10} M$ goes through 0 at $u = 1$ i.e., at $\omega = \frac{1}{T_1}$. If we therefore measure the experimental response of a first-order system and plot it as indicated above, the delay time T_1 is easily estimated by noting the frequency at which a straight line approximation for large values of ω crosses the axis.

Another convenient method of representing $G(i\omega) = Me^{i\theta}$ is directly as a complex vector of magnitude M and argument θ. This is so-called Nyquist diagram. For our first order system $G(i\omega)$ is a semi-circle, starting at 1 for $\omega = 0$, going through $1/(1+i) = \frac{1}{\sqrt{2}}(1-i)$ at $\overset{\omega T_1 = 1}{\cancel{\frac{1}{\sqrt{2}}(1-i)}}$, and ending at 0 for $\omega \to \infty$. For many purposes it is more convenient to plot $1/G(i\omega)$. It will be noted that, in case of a first order system, this results in a straight line, $1/G(i\omega) = 1 + i\omega T_1$





Summerizing the important conclusions for a simple first-order system we note the following:

(1) Its transfer function is $G(p) = 1/(T_1 p + 1)$, which has a simple pole at $p = -1/T_1$.

(2) For a step input, the output approaches the input exponentially with the time constant T_1 .

(3) For a sinusoidal input there is a transient and a steady-state output. The ratio of steady-state output to sinusoidal input, which is most important for stability analysis, may be written down directly from the transfer function

$$[y_o(t)]_{steady} / y_i(t) = M e^{i\theta} = G(i\omega) = \frac{1}{1 + i\omega T_1}$$

(4) The transient disturbance due to the initial condition at $t = 0-\epsilon$ and the transient disturbance associated with the initial application of the force are damped out like e^{-t/T_1} . These terms are important for calculation of the response curve but not for the stability analysis.

Examples of Other First-Order Systems

The following physical systems may be approximated by first order differential equations, and they frequently occur in the synthesis of aircraft control system:

(1) Integrator

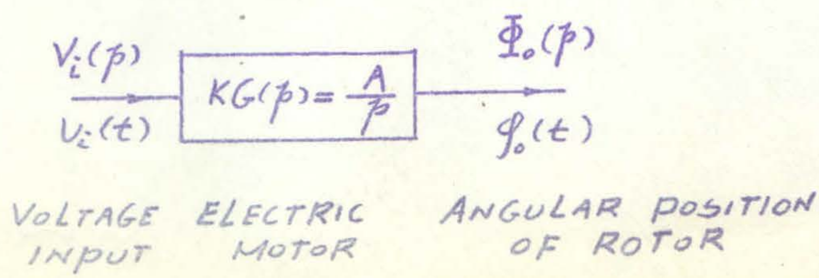
An electric motor whose speed $d\varphi_o/dt$ is proportional to an input voltage V_i follows the equation

$$\frac{d\varphi_o}{dt} = A V_i \qquad (7.33)$$

where A is a scale factor. Thus, the angular rotation φ_o of the motor is proportional to

$$\int_0^t V_i \, dt$$

This relation is represented by the block diagram with Laplace transforms $V_i(p)$.



and $\Phi_o(p)$. This transfer function represents a limiting case of $G(p) = \frac{1}{T_1 p + 1}$ as $T_1 \to \infty$. Thus the transfer function is represented by a pole at the origin. The output response to a sinusoidal input is obtained from

$$M e^{i\theta} = A/i\omega = \left(\frac{A}{\omega}\right) e^{-i\frac{\pi}{2}} = KG(i\omega) \tag{7.34}$$

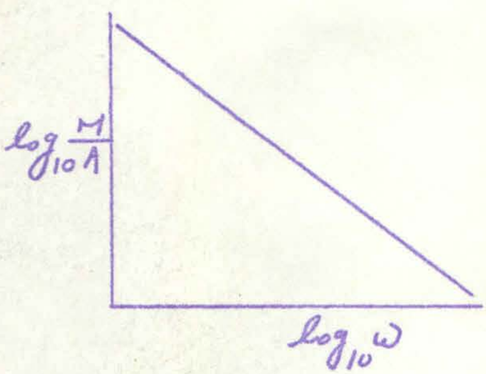
Thus

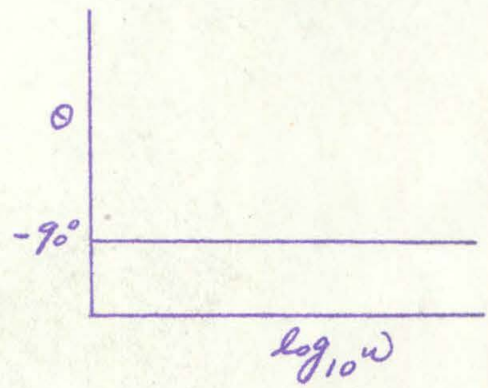
$$M = \frac{A}{\omega} \tag{7.35}$$

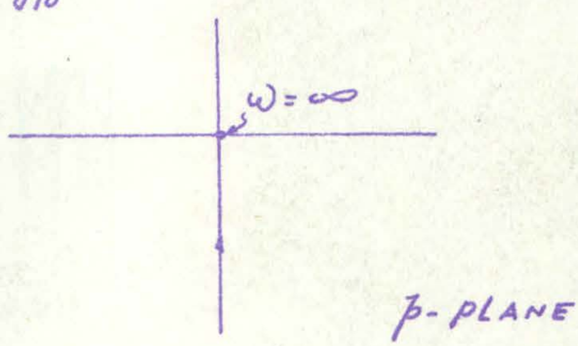
and

$$\theta = -\frac{\pi}{2} \tag{7.36}$$

The output is attenuated at high frequencies, and lags the input by 90°.







(2) Differentiator

A rate gyro gives a voltage output V_o proportional to the angular velocity $d\varphi_i/dt$ of the precession axis, i.e.,

$$V_o = A(d\varphi_i/dt) \tag{7.37}$$

where A is the scale factor. This case is, therefore, the inverse of the preceding one. The transfer function $KG(p) = Ap$ has a zero at the origin. The output response to a sinusoidal input is obtained from

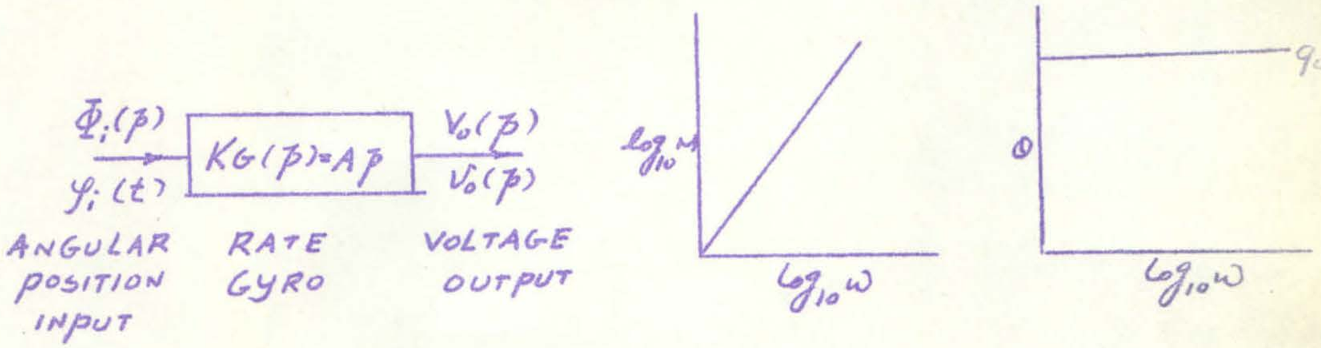
$$Me^{i\theta} = KG(i\omega) = Ai\omega = (A\omega)e^{i\frac{\pi}{2}}$$

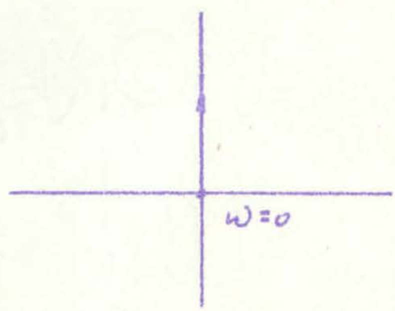
Therefore

$$M = A\omega \tag{7.38}$$

$$\theta = +90^\circ \tag{7.39}$$

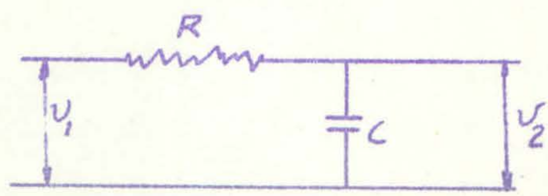
The output is thus proportional to the input frequency and leads by 90°.





(3) Simple lag network

Consider the following circuit:



If j is the current flowing in the resistance and capacitance and if there is no charge in the capacitance at t = 0, we have

$$jR + \frac{1}{C}\int_0^t j(t)\,dt = v_1$$
$$\frac{1}{C}\int_0^t j(t)\,dt = v_2$$

By multiplying these equations by e^{-pt} and integrate from t = o to t = ∞, we have

$$\left(R + \frac{1}{Cp}\right) J(p) = V_1(p)$$
$$\frac{1}{Cp} J(p) = V_2(p)$$

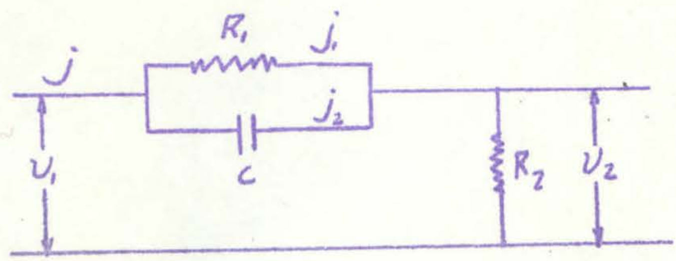
Therefore
$$V_2(p)/V_1(p) = \frac{1}{1+RCp} \tag{7.40}$$

Hence the transfer function of this circuit is the same as the example of spring dash pot system given in the previous system. Only here T_1 = RC. This circuit is frequently used in order to introduce a phase lag into a system. It is generally more convenient to deal with the reciprocal of delay time, and thus we define $\omega_1 = 1/T_1$ and correspondingly the transfer function for a simple lag network

$$G(p) = \frac{1}{1+\left(\frac{p}{\omega_1}\right)} = \frac{\omega_1}{\omega_1 + p}, \quad \omega_1 = 1/T_1 \tag{7.41}$$

(4) Lead Network

Consider the following circuit:



$$j = j_1 + j_2$$

$$R_1 j_1 = \frac{1}{C}\int_0^t j_2(t)\,dt$$

$$v_1 = R_1 j_1 + R_2 j$$

$$v_2 = R_2 j$$

The corresponding Laplace transform equations are

$$J = J_1 + J_2$$

$$R_1 J_1 = \frac{1}{Cp} J_2$$

$$V_1 = R_1 J_1 + R_2 J$$

$$V_2 = R_2 J$$

Therefore

$$\frac{V_2(p)}{V_1(p)} = \frac{R_2 + R_1 R_2 Cp}{(R_1+R_2) + R_1 R_2 Cp} \qquad (7.42)$$

Thence the gain is

$$K = \frac{R_2}{R_1 + R_2} = \varepsilon \qquad (7.43)$$

The value of ε is usually between 0.1 and 1. If we introduce the notation ω_1 as

$$\omega_1 = (R_1 + R_2)\big/ R_1 R_2 C \qquad (7.44)$$

The transfer function can be written as

$$G(p) = \frac{1+(p/\epsilon\omega_1)}{1+(p/\omega_1)} \tag{7.45}$$

(7.45) shows that the transfer function has a zero at $p = -\epsilon\omega_1$, and a pole at $p = -\omega_1$. The characteristics of this system are as follows for a sinusoidal input:

$$Me^{i\theta} = \frac{1}{\epsilon}\,\frac{\epsilon\omega_1 + i\omega}{\omega_1 + i\omega}$$

where
$$M = \sqrt{\frac{1+\frac{1}{\epsilon^2}\left(\frac{\omega}{\omega_1}\right)^2}{1+\left(\frac{\omega}{\omega_1}\right)^2}}, \quad \theta = \tan^{-1}\left(\frac{\omega}{\epsilon\omega_1}\right) - \tan^{-1}\left(\frac{\omega}{\omega_1}\right) \tag{7.46}$$

If we put
$$\frac{1}{\sqrt{\epsilon}}\left(\frac{\omega}{\omega_1}\right) = u$$

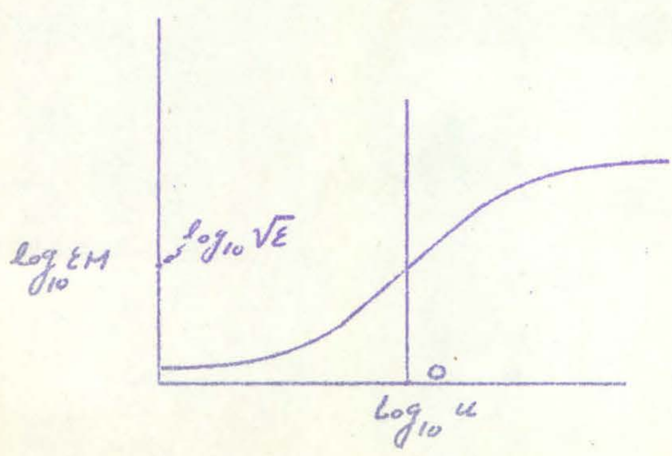
then
$$\sqrt{\epsilon}\,M = \sqrt{\frac{\epsilon + u^2}{1+\epsilon u^2}}, \quad \theta = \tan^{-1}\left(\frac{u}{\sqrt{\epsilon}}\right) - \tan^{-1}(\sqrt{\epsilon}\,u)$$

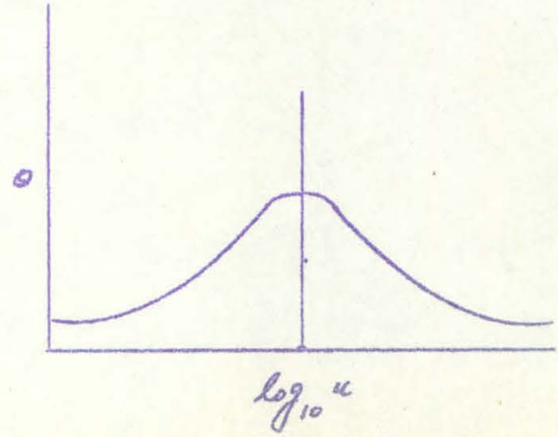
Therefore we have

$$\log_{10}\epsilon M(u) = \log_{10}\sqrt{\epsilon} + \log_{10}\sqrt{\epsilon}\,M(u) = \log_{10}\sqrt{\epsilon} - \log_{10}\sqrt{\epsilon}\,M\left(\tfrac{1}{u}\right)$$

and
$$\theta(u) = \theta\left(\frac{1}{u}\right)$$

Therefore the logarithmic plot of $KMe^{i\theta} = \epsilon Me^{i\theta}$ is as follows:





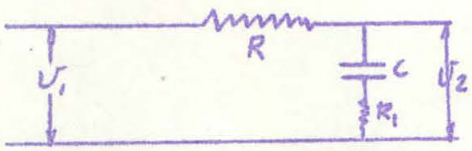
The maximum value of Θ occurs at $u = 1$, and is equal to

$$\Theta_{max} = \tan^{-1}\left(\frac{1}{\sqrt{\varepsilon}}\right) - \tan^{-1}(\sqrt{\varepsilon}) = \frac{\pi}{2} - 2\tan^{-1}(\sqrt{\varepsilon}) \tag{7.47}$$

Therefore this circuit gives a considerable phase lead over a limit band of frequencies. For very large ω, $\varepsilon M = 1$. For very small ω, $\varepsilon M = \varepsilon$.

(5) Restricted lag network:

The following network gives



$$\frac{V_2(p)}{V_1(p)} = \frac{1 + R_1 C p}{1 + (R_1 + R) C p} \tag{7.48}$$

Thus

$$\left.\begin{aligned} K &= 1 \\ G(p) &= \frac{1 + (p/\omega_1)}{1 + (p/\varepsilon\omega_1)} \end{aligned}\right\} \tag{7.49}$$

where $\qquad \omega_1 = 1/RC, \quad \varepsilon = R_1/(R_1 + R)$

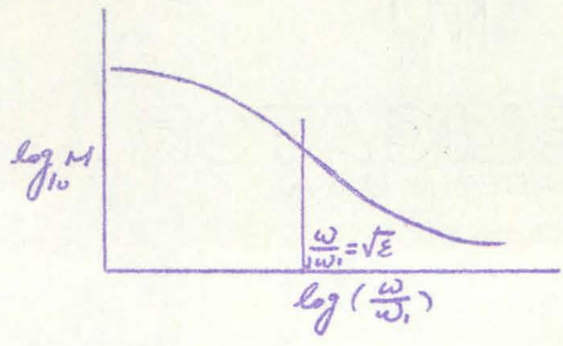
The transfer function $G(p)$ has a zero at $p = -\omega_1$ and a pole at $p = -\varepsilon\omega_1$. The value of ε varies ordinarily between 0.1 and 1.

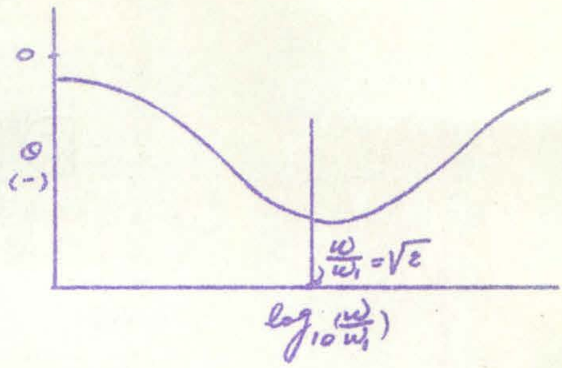
The characteristics of this system are as follows for a sinusoidal input:

$$Me^{i\theta} = G(i\omega) = \frac{1 + i\frac{\omega}{\omega_1}}{1 + i\frac{\omega}{\varepsilon\omega_1}}$$

where $\qquad M = \sqrt{\dfrac{1 + \left(\frac{\omega}{\omega_1}\right)^2}{1 + \left(\frac{\omega}{\varepsilon\omega_1}\right)^2}}, \qquad \theta = \tan^{-1}\dfrac{\omega}{\omega_1} - \tan^{-1}\dfrac{\omega}{\varepsilon\omega_1} \qquad (7.50)$

Therefore the behavior of this network is quite similar to the network studied in the previous section. Only here we have a phase lag for a limited band of frequencies.





(6) Simplified Aerodynamic Stability Equations

The simplified equation for the rolling motion of an airplane is

$$I\frac{d^2\varphi}{dt^2} + L_{\dot\varphi}\frac{d\varphi}{dt} = k\cdot\delta \tag{7.51}$$

where I is the moment of inertia about the roll axis, $L_{\dot\varphi}$ is the aerodynamic damping in roll, and $k\delta$ is the tongue applied due to the aileron deflection δ. Writing $n = \frac{d\varphi}{dt}$,

$$I\left(\frac{dn}{dt}\right) + L_{\dot\varphi} n = k\delta$$

If $n(0) = 0$,

$$(Ip + L_{\dot\varphi})N(p) = k\,D(p)$$

Thus

$$\frac{N(p)}{D(p)} = \frac{k}{Ip + L_{\dot\varphi}}$$

Therefore

$$K = \frac{k}{L_{\dot\varphi}}$$

and

$$G(p) = \frac{1}{1 + \left(\frac{I}{L_{\dot\varphi}}\right)p}$$

The analysis of this simplified case follows that of the simple lag network. Sometimes the aerodynamic damping $L\dot{\varphi}$ is so small as to be negligible, then the analysis follows that of the simple integrator circuit.

Analysis of Second-Order Systems

Let us consider the cantilever spring of a previous section. Attach a finite mass m to the dashpot. Then, the additional force acting upward on the dashpot is $-m\frac{d^2 y_{ob}}{dt^2}$. The equation of motion then becomes

$$m\frac{d^2 y_{ob}}{dt^2} + c\frac{dy_{ob}}{dt} + k\,y_{ob} = k\,y_i \tag{7.52}$$

with the initial conditions

$$\left.\begin{aligned} y_{ob} &= y_b(0) \\ \frac{dy_{ob}}{dt} &= y_b'(0) \end{aligned}\right\} \quad at \quad t = 0-\varepsilon \tag{7.53}$$

We introduce the following ratios

$\omega_0 = \sqrt{k/m}$ = natural frequency of undamped mass-spring system

$\zeta = (c/m)/2\omega_0$ = ratio of actual damping coefficient to critical damping

Then the equation of motion becomes

$$\frac{d^2 y_{ob}}{dt^2} + 2\zeta\omega_0\frac{dy_{ob}}{dt} + \omega_0^2 y_{ob} = \omega_0^2 y_i$$

The Laplace transform equation is

$$(p^2 + 2\zeta\omega_0 p + \omega_0^2)Y_{ob}(p) = \omega_0^2 Y_i(p) + y_b'(0) + (p + 2\zeta\omega_0)y_b(0)$$

Thus

$$\frac{Y_o(p)}{Y_i(p)} = \frac{\omega_0^2}{p^2 + 2\zeta\omega_0 p + \omega_0^2} \tag{7.54}$$

$$Y_b(p) = \frac{y_b'(0) + (p + 2\zeta\omega_0)\, y_b(0)}{p^2 + 2\zeta\omega_0 p + \omega_0^2} \tag{7.55}$$

Thus

$$\left.\begin{aligned} K &= 1 \\ G(p) &= \frac{1}{(\frac{p}{\omega_0})^2 + 2\zeta(\frac{p}{\omega_0}) + 1} \end{aligned}\right\} \tag{7.56}$$

This transfer function has no zeros, but it has two poles at p_1 and p_2, the roots of the denominator

$$(\frac{p}{\omega_0})^2 + 2\zeta(\frac{p}{\omega_0}) + 1 = 0$$

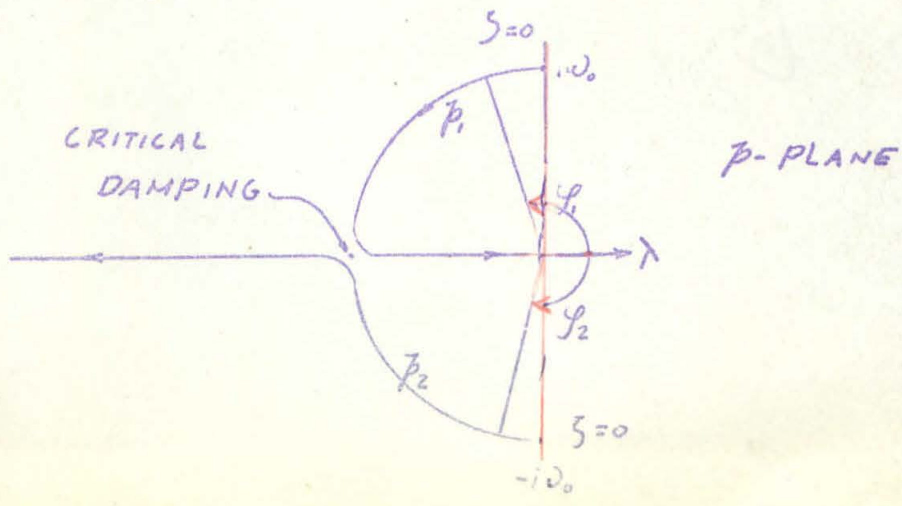
Or

$$\left.\begin{aligned} \frac{p_1}{\omega_0} &= -\zeta + \sqrt{\zeta^2 - 1} \\ \frac{p_2}{\omega_0} &= -\zeta - \sqrt{\zeta^2 - 1} \end{aligned}\right\} \quad \zeta^2 > 1 \tag{7.57}$$

And

$$\left.\begin{aligned} \frac{p_1}{\omega_0} &= -\zeta + i\sqrt{1-\zeta^2} = \frac{1}{\omega_0}(\lambda + i\nu) = e^{i\varphi_1} \\ \frac{p_2}{\omega_0} &= -\zeta - i\sqrt{1-\zeta^2} = \frac{1}{\omega_0}(\lambda - i\nu) = e^{-i\varphi_1} \end{aligned}\right\} \zeta^2 < 1 \tag{7.58}$$

These values of p_1 and p_2 can be plotted on the complex p-plane with $0 < \zeta < \infty$ as parameter. When $\zeta = 0$, $\varphi_1 = \frac{\pi}{2}$. As ζ increases, φ_1 also increases till $\zeta = 1$, $\varphi_1 = \pi$. Further increase in ζ, moves p_1 towards the origin along the negative part of the real axis; while p_2 moves towards negative infinity.



The solution $y_b(t)$ due to the initial conditions is found as follows for the case $\zeta < 1$. We introduce $\lambda = -\zeta\omega_0$ and $\nu = \sqrt{1-\zeta^2}\,\omega_0$. Then it can be easily found that

$$y_b(t) = \frac{y_b'(0)}{\nu} e^{\lambda t}\sin\nu t + y_b(0)\, e^{\lambda t}\cos\nu t + \left(\frac{-\lambda}{\nu}\right) y_b(0)\, e^{\lambda t}\sin\nu t \tag{7.59}$$

Since λ is a negative number, it is seen that the initial motion due to the boundary condition is a damped sine wave with damping factor $\lambda = -\zeta\omega_0$ and with the natural frequency $\nu = \sqrt{1-\zeta^2}\,\omega_0$.

Let us examine more carefully the forced motion due to the input $y_i(t)$.

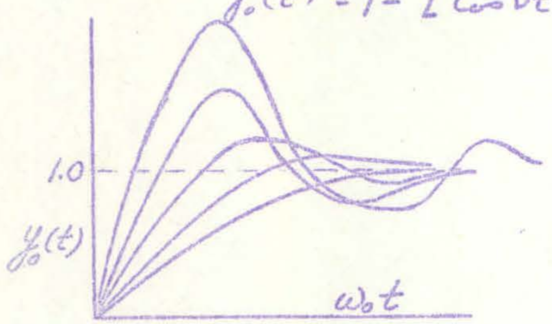
(1) Special case: $y_i(t) = h(t)$

For the special case of a unit step input in displacement, $y_i(t) = h(t)$ and $Y_i(p) = \frac{1}{p}$. The corresponding solution for the second-order equation is then

$$Y_0(p) = \frac{1}{p}\,\frac{\omega_0^2}{(p-\lambda)^2+\nu^2} \tag{7.60}$$

and thus

$$y_0(t) = 1 - \left[\cos\nu t - (\lambda/\nu)\sin\nu t\right] e^{\lambda t} \tag{7.61}$$



The region $0.4 < \zeta < 1$ is ordinarily considered a good compromise between quick approach to the input and excessive overshooting.

(2) Special Case: $y_i(t) = y_{im}\, e^{i\omega t}$

The special case of sinusoidal input gives

$$Y_i(p) = y_{im}\frac{1}{p-i\omega} \tag{7.62}$$

So

$$Y_0(t) = y_{im}\frac{1}{p-i\omega}\,\frac{\omega_0^2}{p^2+2\zeta\omega_0 p+\omega_0^2} \tag{7.63}$$

Therefore

$$y_o(t) = y_{im}\,G(i\omega)\,e^{i\omega t} + y_{im}\left[\frac{\omega_0^2}{\lambda+i(\nu-\omega)}\cdot\frac{1}{2i\nu}\right]e^{(\lambda+i\nu)t} + y_{im}\left[\frac{\omega_0^2}{\lambda-i(\nu-\omega)}\;\frac{-1}{2i\nu}\right]e^{(\lambda-i\nu)t} \qquad (7.64)$$

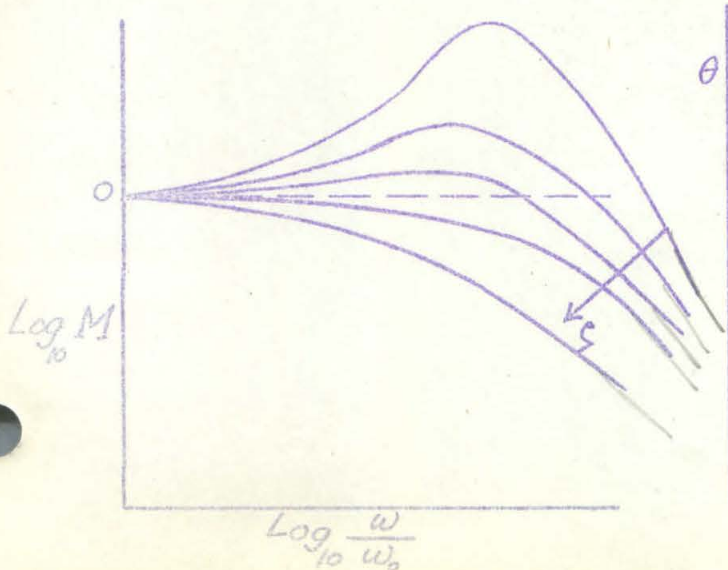
The first part is the steady state output and the remainder is the transcient which will be damped out as $t \to \infty$. Here again

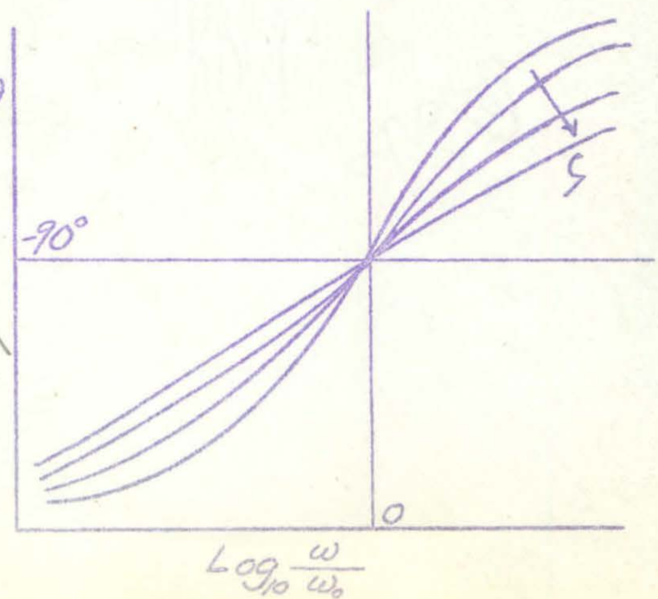
$$\left[y_o(t)\right]_{steady} \Big/ y_i(t) = Me^{i\theta} = G(i\omega)$$

with
$$Me^{i\theta} = \frac{\omega_0^2}{(\omega_0^2-\omega^2)+2i\zeta\omega_0\omega}$$

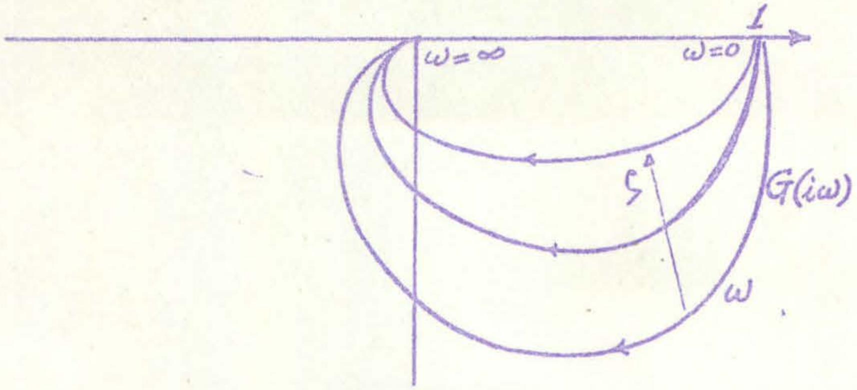
Therefore
$$\left.\begin{aligned} M &= \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2+\left(2\zeta\frac{\omega}{\omega_0}\right)^2}} \\ \text{and}\qquad \tan\theta &= -2\zeta(\omega/\omega_0)\Big/\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right] \end{aligned}\right\} \qquad (7.65)$$

The maximum of M occur near $\omega/\omega_0 \doteq 1$, where $M = \frac{1}{2\zeta}$, and $\theta = -90°$. As $\omega/\omega_0 \to \infty$, $\theta \to -180°$ and $M \sim 1\big/\left(\frac{\omega}{\omega_0}\right)^2$ or $\log_{10} M \cong -2\log_{10}\left(\frac{\omega}{\omega_0}\right)$. The acoustic engineer will call this a slope of -12.04 lb. per octave.





The transfer function $G(i\omega) = M e^{i\theta}$ is easily plotted on the basis of the previous calculations. The vector $G(i\omega)$ starts from $M = 1$, $\theta = 0$ When $\omega = 0$; the vector increases to a maximum at resonance and then decreases in magnitude again as $M \to 0$ and $\theta \to -180^\circ$ as $\omega \to \infty$.



(3) <u>Other Second-Order Systems</u>

The following physical systems that occur in the automatic control of aircraft may be approximated by second-order systems.

a) Mechanical Spring-mass-damping systems

1) Hydraulic servo system

2) Rate gyro - A better approximation than our previously used equation $KG(p) = Ap$ is

$$KG(p) = \frac{Ap}{\left(\frac{p}{\omega_0}\right)^2 + 2\zeta\left(\frac{p}{\omega_0}\right) + 1}$$

3) Accelerometer

$$KG(p) = \frac{Ap^2}{\left(\frac{p}{\omega_0}\right)^2 + 2\zeta\left(\frac{p}{\omega_0}\right) + 1}$$

4) Electric Motor

$$KG(p) = \frac{A}{p\left\{p + \left(\frac{1}{T}\right)\right\}}$$

b) Electrical Networks

c) Aerodynamic systems

Composition of Elements

Consider a series of arrangements of n elements. The r th element has a gain K_r, and a transfer function $G_r(p) = M_r e^{i\theta_r}$. This series array is equivalent to a single block having a transfer function $KG(p)$ where

$$K = K_1 \, K_2 \cdots K_r \cdots K_n \tag{7.66}$$

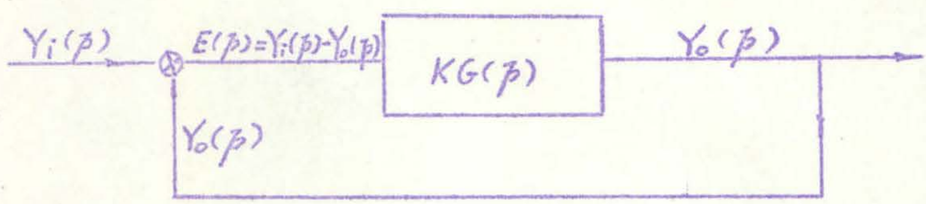
$$\begin{aligned} G(p) &= (M_1 e^{i\theta_1})(M_2 e^{i\theta_2}) \cdots (M_r e^{i\theta_r}) \cdots (M_n e^{i\theta_n}) \\ &= (M_1 M_2 \cdots M_r \cdots M_n)\, e^{i(\theta_1+\theta_2+\cdots+\theta_r+\cdots+\theta_n)} \\ &= M e^{i\theta} \end{aligned}$$

Therefore

$$\left.\begin{aligned} \log_{10} M &= \log_{10} M_1 + \log_{10} M_2 + \cdots + \log_{10} M_r + \cdots + \log_{10} M_n \\ \theta &= \theta_1 + \theta_2 + \cdots + \theta_r + \cdots + \theta_n \end{aligned}\right\} \tag{7.67}$$

This shows the usefulness of the logrithmic M.

A simple feedback servo is shown schematically as follows:



The equations that apply are

$$E(p) = Y_i(p) - Y_o(p) \tag{7.68}$$

$$Y_o(p) = KG(p)\,E(p) \tag{7.69}$$

Thus

$$\frac{Y_o(p)}{E(p)} = KG(p) \qquad \frac{E(p)}{Y_o(p)} = \frac{1}{KG(p)} = [KG(p)]^{-1} \tag{7.70}$$

And $$\frac{Y_i(p)}{E(p)} = \frac{E(p)+Y_o(p)}{E(p)} = 1 + KG(p) \tag{7.71}$$

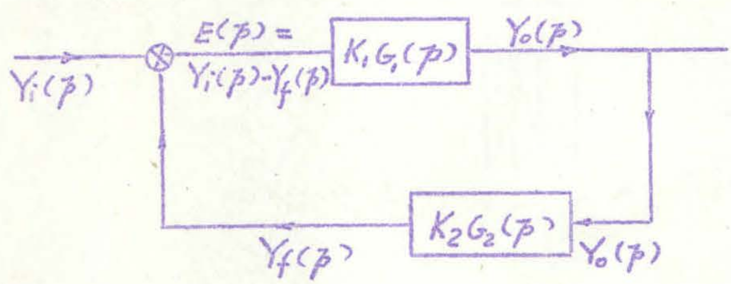
The static case corresponds to zero frequency. Thus (7.70) shows that the ratio of error to input at zero frequency a static condition is smaller for larger values of the gain K. Therefore in order to have an accurate control system with small "steady state" errors, the gain K should be large. This is one of the design condition.

Furthermore $$\frac{Y_o(p)}{Y_i(p)} = \frac{KG(p)}{1+KG(p)}\,; \quad \frac{Y_i(p)}{Y_o(p)} = 1 + [KG(p)]^{-1} \tag{7.72}$$

The overall ratio $Y_o(p)/Y_i(p)$ could be represented by a system transfer function $K_S G_S(p)$.

The feedback servo is the most important element for automatic control systems. The input $Y_i(p)$ is compared with the actual output $Y_o(p)$, and only the difference $E(p) = Y_i(p) - Y_o(p)$ is passed on to the airplane as a control signal. This trick permits the output to be controlled by an error signal rather than by direct input, and as a result, the accuracy and speed of response are much improved. We shall see this presently.

The more general feedback servo is as follows:



Thus its arrangement is similar to the simple servo, except that the signal that is fed back is modified in amplitude and phase by a corrective network with the transfer function $K_2 G_2(p)$. Hence

$$\left.\begin{aligned} E(p) &= Y_i(p) - Y_f(p) \\ Y_f(p) &= K_2 G_2(p) Y_o(p) \\ Y_o(p) &= K_1 G_1(p) E(p) \end{aligned}\right\} \tag{7.73}$$

Therefore

$$\frac{Y_o(p)}{Y_i(p)} = \frac{K_1 G_1(p)}{1 + K_1 K_2 G_1(p) G_2(p)} \qquad (7.74)$$

Appendix to Chapter 7

As a simple example of feedback control, let us consider a turbo-generator set. The angular speed of the set is determined by the balance of the driving torque of the steam turbine and the torque absorbed by the alternator. The torque absorbed by the alternator is in turn determined by the electrical load. The ideal operation is then one where the steam inlet valve to the turbine is so manipulated that these two torques are exactly equal. Practically, this ideal balance can never be achieved and we have always the residual error torque $\delta(t)$. If I is the moment of inertia of the rotating parts, ω' the deviation of the angular velocity from the desired value, say, 3600 rpm. and C is the "damping coefficient", then

$$I \frac{d\omega'}{dt} + C\omega' = \delta(t) \tag{A-1}$$

This is the fundamental equation for open-cycle control. Since I is very large, C is very small. The characteristic time of the system is very large, being I/C. Furthermore the steady state error, being δ/C will be also rather large.

In the language of servo-mechanism, (A-1) is represented as

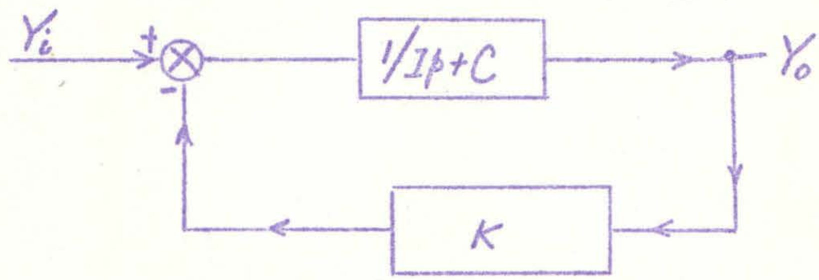
$$Y_i \longrightarrow \boxed{1/Ip+C} \longrightarrow Y_o$$

where $Y_i(p)$ is the Laplace transform of $\delta(t)$ and $Y_o(p)$ is the Laplace transform of the $\omega'(t)$.

Now we can change the control system to the efficient closed cycle control by simply letting the steam turbine valve be opened or closed by an additional amount according to the ω', the deviation in angular velocity. That is, we continuously measure ω' and let the torque of the steam turbine be increased by an amount $-K\omega'$. Then the dynamic equation correspond to (A-1) is

$$I\frac{d\omega'}{dt} + C\omega' = \delta(t) - K\omega' \tag{A-2}$$

The characteristic time of the system is now $I/(C + K)$, and the steady state error is $\delta/(C + K)$. By making K large, both of these equantities can be made to be small. We have then a control system of rapid response and great accuracy. In a diagram, the system is



Therefore the closed-cycle control is very much superior in comparison with the open-cycle control.

8. Stability of Feedback Control Systems

We shall give an outline of the various methods used to determine the performance of the feedback control systems. The treatment still follows the paper by W. Bollay.

Routh Criterion for Coefficients of the Frequency Equation

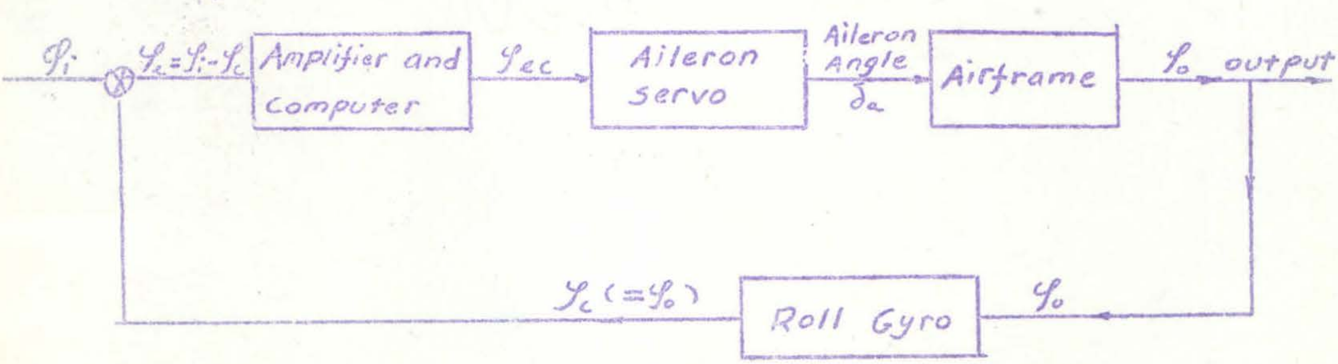
We have seen that for the simple feedback network the overall system transfer function is defined by

$$\frac{Y_o(p)}{Y_i(p)} = K_s G_s(p) = \frac{N_s(p)}{D_s(p)} = \frac{KG(p)}{1+KG(p)} \tag{8.1}$$

where $N_s(p)$ and $D_s(p)$ are polynomials. The transfer function $G(p)$ will ordinarily be expressible as the ratio of two polynomials, each of which is given in factored form. Thus for the following roll control system, the transfer function is

$$KG(p) = \frac{N(p)}{D(p)} = \left\{\frac{k_1/L_{\dot\varphi}}{p(1+pT_1)}\right\}\left\{\frac{1+\varepsilon_2 T_2 p}{1+T_2 p}\,\frac{\varepsilon_3+T_3 p}{1+T_3 p}\cdot\frac{\varepsilon_4+T_4 p}{1+T_4 p}\right\}\left\{\frac{\mu\omega_o^2}{p^2+2\zeta\omega_o p+\omega_o^2}\right\} \tag{8.2}$$

where the first part is due to airplane, the second part due to amplifier and computer and the third part due to aileron control servo.



The numerator $N(p)$ of this function is of the third degree in p , the denominator $D(p)$ is of the seventh degree in p . Each is given in factored form so that we know the three zeros and seven poles of $G(p)$. We note that

$$K_S G_S(p) = \frac{N_S(p)}{D_S(p)} = \frac{N(p)}{D(p)+N(p)} \tag{8.3}$$

Thus the zeros of $G_S(p)$ are the same as those of $G(p)$. The poles of $G_S(p)$ are, however, changed. The degree of $D_S(p)$ is the same as that of $D(p)$, since ordinarily $D(p)$ is of equal or higher degree than $N(p)$. Let us write, in general

$$K_S G_S(p) = \frac{N_S(p)}{D_S(p)} = A\frac{(p-z_1)(p-z_2)----(p-z_m)}{(p-p_1)(p-p_2)---(p-p_n)} \tag{8.4}$$

where $z_1, z_2, ----z_m$ are the zeros of $G_S(p)$, $p_1, p_2 -----p_n$ are poles of $G_S(p)$ and A is a constant.

The principal mathematical problem of the servo analysis is ordinarily to go from the transfer function $G(p)$ to the function $G_S(p)$ in factored form. For after the latter has been written in factored form, the solution of the problem is immediately available. If, for example, the input is a sinusoidal force $F_i = F_{max.} e^{i\omega t}$, then

$$Y_i(p) = F_{max.} 1 \cancel{p}/(p-i\omega) \tag{8.5}$$

and the solution is

$$y_o(t) = F_{max.}\left[K_S G_S(i\omega)e^{i\omega t} + \sum_{r=1}^{n} \frac{N_S(p_r)}{(p_r-i\omega)\left(\frac{dD_S}{dp}\right)_{p=p_r}} e^{p_r t}\right] \tag{8.6}$$

where p_r's are the roots of the so-called frequency equation $D_s(p) = 0$. It is obvious that in order for the terms $e^{p_r t}$ to vanish as $t \to \infty$, it is necessary that the real parts of p_r's be all negative. This means that the roots p_r must be all in the negative half of the p-plane. The determination of the roots p_r is ordinarily not simple if the equation $D_s(p)$ is a higher-order polynomial. The usual classical technique to assume the negativeness of the real parts of p_r is a series of inequalities about the coefficients of the polynomial $D_s(p)$. It has the disadvantage that one loses all insight into the physical behavior as the various parameters are changed. The following techniques are superior in this respect.

Root Locus Method of W. R. Evans

Let us consider the function

$$K_s G_s(p) = \frac{KG(p)}{1+KG(p)} = \frac{N_s(p)}{D_s(p)}$$

and studies the variation of the poles of $G_s(p)$ i.e., the roots of the frequency equation $D_s(p) = 0$ as the parameter K varies from 0 to ∞. Let us assume that $G(p)$ is given in factored form

$$G(p) = A\,\frac{(p-s_1)(p-s_2)----(p-s_m)}{(p-\delta_1)(p-\delta_2)----(p-\delta_n)} \tag{8.7}$$

where

$$A = \frac{(-\delta_1)(-\delta_2)----(-\delta_n)}{(-s_1)\ (-s_2)\ ---\cdot(-s_m)} \tag{8.8}$$

Generally the denominator of $G(p)$ is of equal or higher order than the denominator i.e., $n \geq m$. Let us express each of the factors in vector form:

$$\begin{aligned}
p-s_1 &= S_1 e^{i\varphi_1}\\
p-s_2 &= S_2 e^{i\varphi_2}\\
&-----\\
p-s_m &= S_m e^{i\varphi_m}\\
p-\delta_1 &= Q_1 e^{i\Theta_1}\\
&-----\\
p-\delta_n &= Q_n e^{i\Theta_n}
\end{aligned} \tag{8.9}$$

The vector $S_r e^{i\varphi_r}$ starts from the end point of S_r and goes to the end point of p. The vector $Q_r e^{i\theta_r}$ starts from the end point of q_r and goes to the end point of p. The points S_r and q_r are, of course, the zeros and poles of $G(p)$. The point p is a variable point in the p-plane. Thus

$$G(p) = A\,\frac{(S_1 e^{i\varphi_1})(S_2 e^{i\varphi_2}) \cdots (S_m e^{i\varphi_m})}{(Q_1 e^{i\theta_1})(Q_2 e^{i\theta_2}) \cdots (Q_n e^{i\theta_n})}$$

Now since the physical constants of any unit of the system are real, the coefficients of $N(p)$ and $D(p)$ are real. Moreover the roots of $N(p)$ and $D(p)$ have generally negative real parts. Therefore A is real and positive. Then

$$G(p) = A\,\frac{S_1 S_2 \cdots S_m}{Q_1 Q_2 \cdots Q_n}\, e^{i[(\varphi_1+\varphi_2+\cdots+\varphi_m)-(\theta_1+\theta_2+\cdots\theta_n)]}$$

Therefore if we write

$$G(p) = R e^{i\theta} \tag{8.10}$$

then

$$\left.\begin{aligned} R &= A(S_1 S_2 \cdots S_m / Q_1 Q_2 \cdots Q_n) \\ \text{and}\quad \theta &= (\varphi_1+\varphi_2+\cdots\varphi_m)-(\theta_1+\theta_2+\cdots\theta_n) \end{aligned}\right\} \tag{8.11}$$

The poles of $G_s(p)$ are found from the condition that

$$1 + KG(p) = 0$$

Or

$$KRe^{i\theta} = e^{i\pi}$$

Hence the conditions for the pole p_r of $G_s(p)$ are

$$\left.\begin{aligned} &KR = 1 \\ \text{and} \quad &\theta = \pi \pm 2\pi\cdot \text{integer}) \end{aligned}\right\} \qquad (8.12)$$

Evans studies this problem in two steps. He determines first of all the locus of all points which satisfy the angle condition. Then he computes the value of K for various points along this locus by measuring the lengths $S_1\ S_2 --- S_m$ and $Q_1\ Q_2 --- Q_n$ and computing the ratio R. For a given value of K, the points $p_1, p_2 ---- p_n$ along these loci are then the roots of $D_s(p) = 0$, i.e., the poles of $G_s(p)$.

Evans has developed a number of simple rules that determine the locus of points satisfying the angle condition.

Rule 1 --- For a gain $K = 0$, in order for $1 + KG(p) = 0$, $G(p)$ must be very large, i.e., p must be a pole of $G(p)$. Therefore for $K = 0$, the poles of $G_s(p)$ are poles of $G(p)$.

Rule 2 --- For $K \to \infty$, in order for $1 + KG(p) = 0$, $G(p)$ must be very small. Therefore for $K \to \infty$, the poles of $G_s(p)$ could be zeros of $G(p)$. But if $n > m$, the number of zeros of $G(p)$ is less than the number of poles of $G_s(p)$. However in that case $G(p) \to 0$ as $p \to \infty$. Therefore the missing poles are supplied by $p = \infty$. Furthermore, since for very large p,

$$G(p) \sim \frac{1}{p^{n-m}}$$

Thus the condition $1 + KG(p) = 0$ requires

$$p^{n-m} = Ce^{i\pi}$$

Hence the asymptotes of p_r have the phase angles

$$\frac{\pi}{n-m} \pm \frac{2k\pi}{n-m}, \qquad k = 1, 2, 3 \cdots$$

In other words, when $n - m = 1$, we have one asymptote with phase angle π, When $n - m = 2$, we have two asymptotes with phase angle $\pi/2$ and $\frac{3\pi}{2}$. When $n - m = 3$, we have three asymptotes with angle $\pi/3$, π, $-\pi/3$. When $n - m = 4$, we have four asymptotes with angles $\pi/4$, $3\pi/4$, $-3\pi/4$, $-\pi/4$.

Rule 3 -- The root locus along the real axis is along alternate segments connecting the zeros and poles located on the real axis, starting with the one farthest to the right.

This rule may be easily verified by considering any point on the real axis. The angle to this point from a pair of conjngate complex poles or zeros is $+\theta$ and $-\theta$, respectively, and thus their sum is zero. The angle to this point from the poles or zeros on the axis is $\theta = 0$ for all poles or zeros to the left of p, and it is $\theta = 180^\circ$ for all poles or zeros to the right of p. Thus, the sum is 180° if there is an odd number of poles and zeros to the right of p.

Rule 4 -- If there is a breakaway of the root locus from the real axis, the point of breakaway may be estimated from the condition that, for a small displacement $\Delta\omega$ from the axis, the increase in angle due to the poles and zeros on the axis to the left must be just balanced by the effect of these to the left.

Example

$$G(p) = \frac{(0.001)(2)(6)}{(p+0.001)(p+2)(p+6)}$$

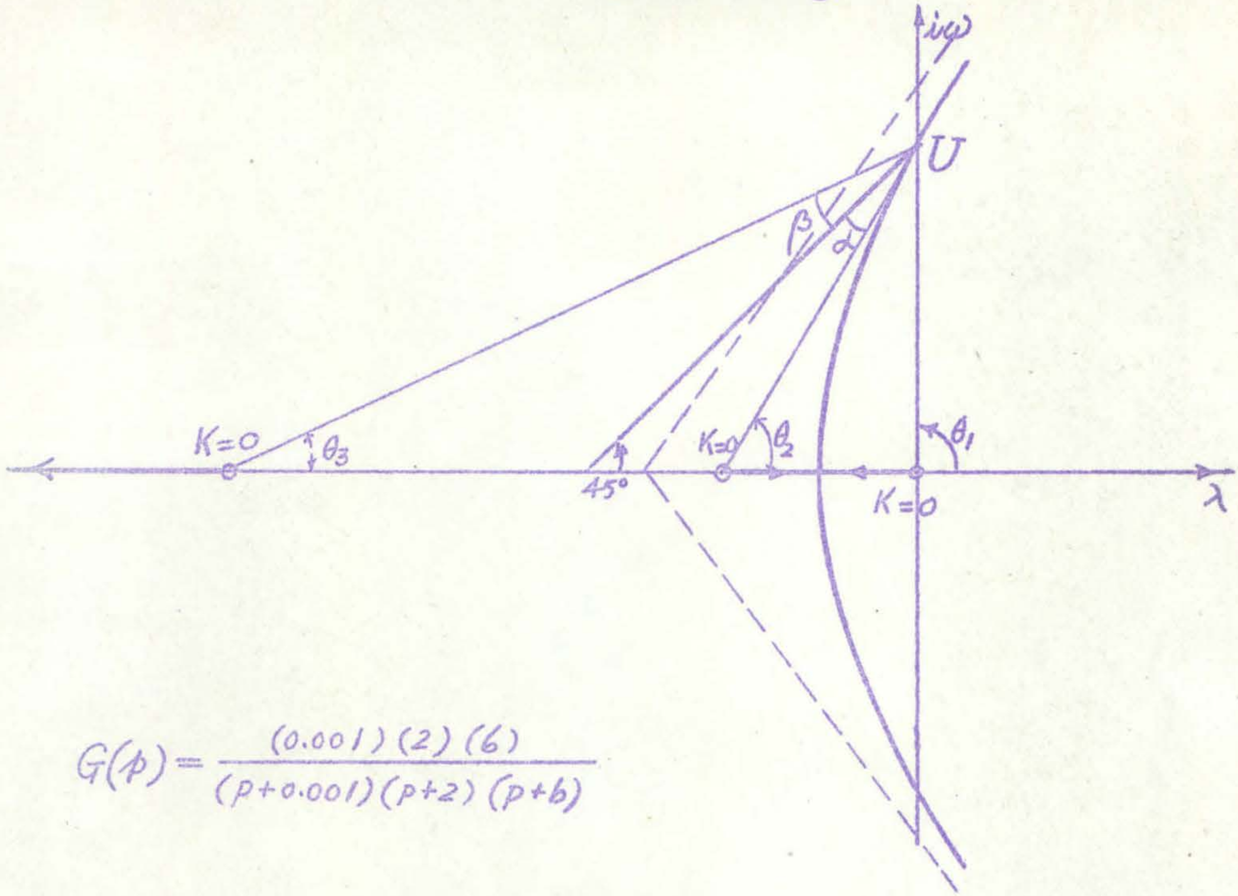
At $K = 0$, loci start from -0.001, -2, and -6. Section of loci between -0.001 and -2, between -6 and $-\infty$. Asymptotes $+60^\circ$, -60°, 180°. Breakaway from λ - axis at λ_1 between -0.001 and -2.

$$\frac{\Delta\omega}{\lambda_1 + 0.001} + \frac{\Delta\omega}{\lambda_1 + 2} + \frac{\Delta\omega}{\lambda_1 + 6} = 0$$

Or $$(\lambda_1+2)(\lambda_1+6)+(\lambda_1+0.001)(\lambda_1+6)+(\lambda_1+0.001)(\lambda_1+2)=0$$

Hence $$3\lambda_1^2+16.002\lambda_1+12.008=0$$

$$\lambda_1=-\frac{16.002}{6}-\sqrt{\left(\frac{16.002}{6}\right)^2-\frac{12.008}{3}}=-.904$$



$$G(p)=\frac{(0.001)(2)(6)}{(p+0.001)(p+2)(p+6)}$$

Rule 5 --- The point U at which the root locus crosses the imaginary axis into the positive half-plane can often be estimated by taking advantage of the properties of the right angle.

Example

$$G(p)=\frac{(0.001)(2)(6)}{(p+0.001)(p+2)(p+6)}\simeq\frac{(.001)(2)(6)}{p(p+2)(p+6)}$$

the approximation is valid

For p not near the origin. Then $\Theta = -180^\circ = -\Theta_2 - \Theta_3 - \Theta_1$

But $\Theta_1 \cong 90^\circ$ So

$$\Theta_2 + \Theta_3 = 90^\circ$$

But

$$\left.\begin{aligned} 45^\circ &= \Theta_3 + \beta \\ 45^\circ + \alpha &= \Theta_2 \end{aligned}\right\} \qquad \begin{aligned} 90^\circ + \alpha &= \beta + \Theta_2 + \Theta_3 \\ \therefore \alpha &= \beta \end{aligned}$$

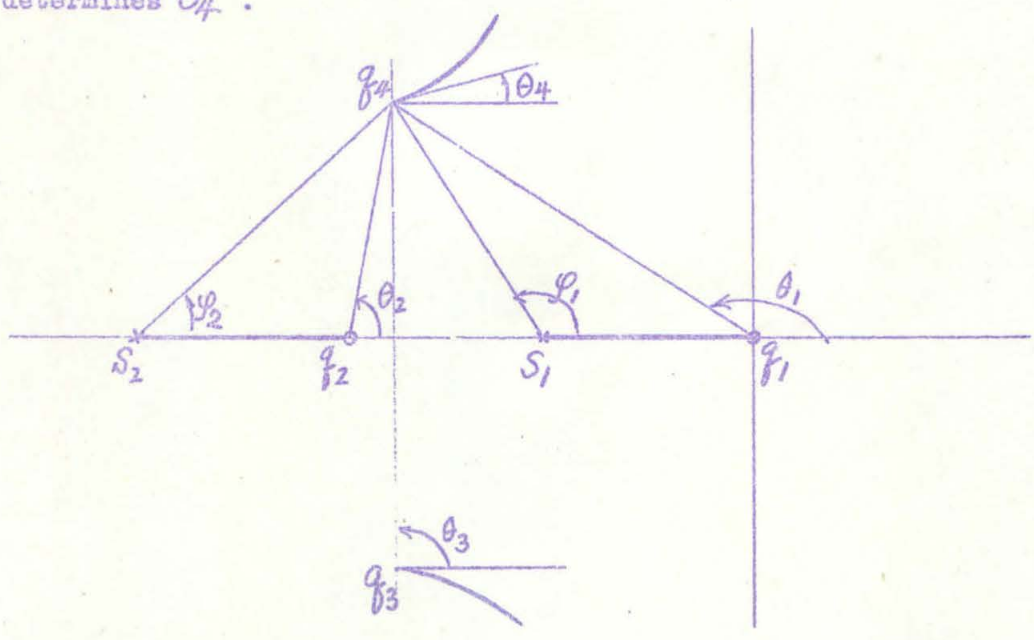
This is the geometrical condition for determining $\mathcal{U}$.

Rule 6 --- The direction of locus departure from a pole (or locus approach to a zero) may be easily estimated by computing the angle at the pole under consideration from all of the other poles and zeros in the field.

Example The following figure indicates that for small displacements away from the pole q_4, the angles φ_1, φ_2 : $\Theta_1 . \Theta_2 . \Theta_3$ from other zeros and poles remain constant. Thus the angle Θ_4 is given by

$$(\varphi_1 + \varphi_2) - [(\Theta_1 + \Theta_2 + \Theta_3) + \Theta_4] = 180^\circ$$

This determines Θ_4.



These rules give the essential characteristics of the root loci. For intermediate locations, the root locus is found by taking a number of trial points. The value of K can then be calculated along the interested parts of the loci. The desired K then can be ascertained.

Hydrodynamic Analogy

The condition for the pole p_r of $G_s(p)$ is

$$1 + KG(p) = 0$$

Or using (8.7)

$$\log A + \sum_{r=1}^{m} \log(p - s_r) - \sum_{r=1}^{n} \log(p - g_r) = \log(\tfrac{1}{K}) - i\pi$$

Therefore

$$-\frac{1}{2\pi}\sum_{r=1}^{m} \log(p - s_r) + \frac{1}{2\pi}\sum_{r=1}^{n} \log(p - g_r) = \frac{1}{2\pi}\log(AK) + i\frac{1}{2} \quad (8.13)$$

If we consider p to be the complex coordinate, then the left side of (8.13) represents the complex potential function $(\phi - i\psi)$ for a group of m sinks of strength unity located at s_r and a group of n sources of the same strength located at g_r. Therefore there is net source of strength $n - m$. (8.13) can be interpreted then as the equation for the streamline of $\psi = \frac{1}{2}$. In other words, the root loci of Evans are the branches of $\frac{1}{2}$ – streamline in the complex p – plane. The value of the potential function along the streamline when multiplied by 2 π gives the logarithm of AK.

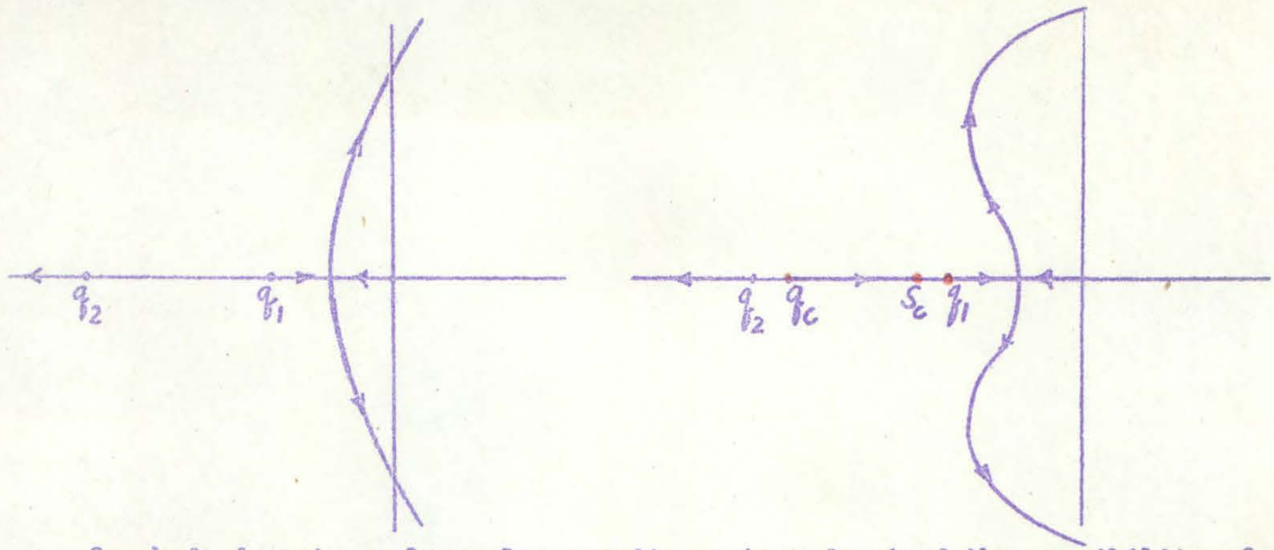
With this concept of hydrodynamic analogy, the general character of the root locus can be sketched immediately. Furthermore it is also a useful ~~desire~~ device to suggest modifications of the system to achieve better system performance. For instance, the system characterized by

$$G(p) = \frac{g_1 g_2}{p(p - g_1)(p - g_2)}$$

may have the disadvantage of being unstable in the closed-loop performance at too low values of gain K. Graphically, it immediately suggests putting a sink at s_c near to q_1 and a source near to q_2. Thus

$$G(p) = \frac{q_c}{s_c}\,\frac{q_1 q_2}{p(p-q_1)(p-q_2)}\,\frac{p-s_c}{p-q_c}$$

Since $(-q_c) > (-s_c)$, this additional series network is a lead circuit discussed in the previous chapter.



Our hydrodynamic analogy also permits us to understand the possibility of speed up the response of a slow mechanism by using the feedback system. We note first that the scale of the p -plane is the inverse of time scale. Thus a small magnitude of p_r means slow response. Now suppose we have a linear mechanical system of first order characterized by a small q_1 on the negative part of the real axis in the p -plane. Putting this mechanical system in series with a fast damped electric circuit with a large q_2 will not improve matters because we still have the q_1 pole. But if we now introduce the feedback, then the roots p_1 and p_2 will lie between q_1 and q_2, according to our hydrodynamic analogy. This means $|p_1| >> |q_1|$ and the response time of the feedback system is very much shorter than the open system without feedback. In fact, it is not difficult to see that feedback system allows us to control the response characteristics of the overall system at will by putting a appropriate circuit in series to the orginal system.

Nyquist and Bode Diagrams

Routh deduced his famous criterion for stability of linear vibrating systems by using the Cauchy Theorem* in the theory of complex variables. This theorem states that if $f(p)$ is the vector representing the function of a complex variable p, this function $f(p)$ having s poles and r zeros within a closed contour C of the p-plane, and as the point p moves around the closed contour C encircling it once in the clockwise direction, the vector $f(p)$ carries out $r-s$ clockwise revolutions about the orgin. Routh applied this theorem to find the number of roots of the frequency equation $D_s(p)=0$ in the right half of the p- plane. He took as boundaries for the contour C: a) the imaginary axis, $p=i\omega$[along this axis $D_s(p)=D_s(i\omega)$] ; and b) the infinite half circle to the right, $p=Re^{i\phi}$, where $R \rightarrow \infty$. Nyquist carried over the application of these cirteria to the simple feedback network for which

$$K_sG_s(p)=\frac{N_s(p)}{D_s(p)}=\frac{KG(p)}{1+KG(p)}$$

Or

$$\frac{1}{K_sG_s(p)}=\frac{D_s(p)}{N_s(p)}=\frac{1}{KG(p)}+1$$

The point $[D_s(p)]/[N_s(p)]=0$ corresponds to $\frac{1}{KG(p)}=-1$ or $\frac{1}{G(p)}=-K$. Thus, if the vector $[D_s(p)]/[N_s(p)]$ encircles the origin as the point p moves clockwise around the contour C of the right half p- plane, then the vector $1/[KG(p)]$ encircles the point -1, or the vector $1/[G(p)]$ encircles the point - K.

Let us illustrate the application of this method for the simple transfer function $G(p)=\frac{1}{p(1+pT_1)(1+pT_2)}$

* See Cf. Whittaker and Watson, "Modern Analysis" p.119, § 6.31 (4th Edition)

and thus

$$\frac{1}{G(p)} = p(1+pT_1)(1+pT_2)$$

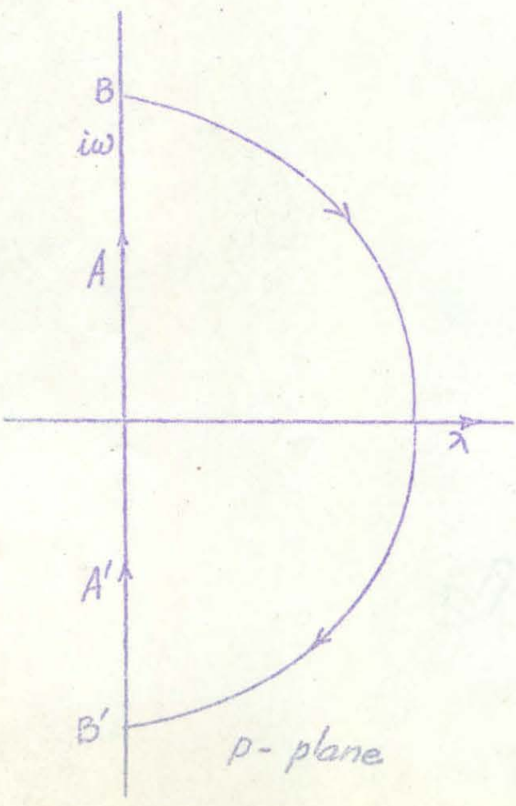
(a) Along the imaginary axis, $p = i\omega$

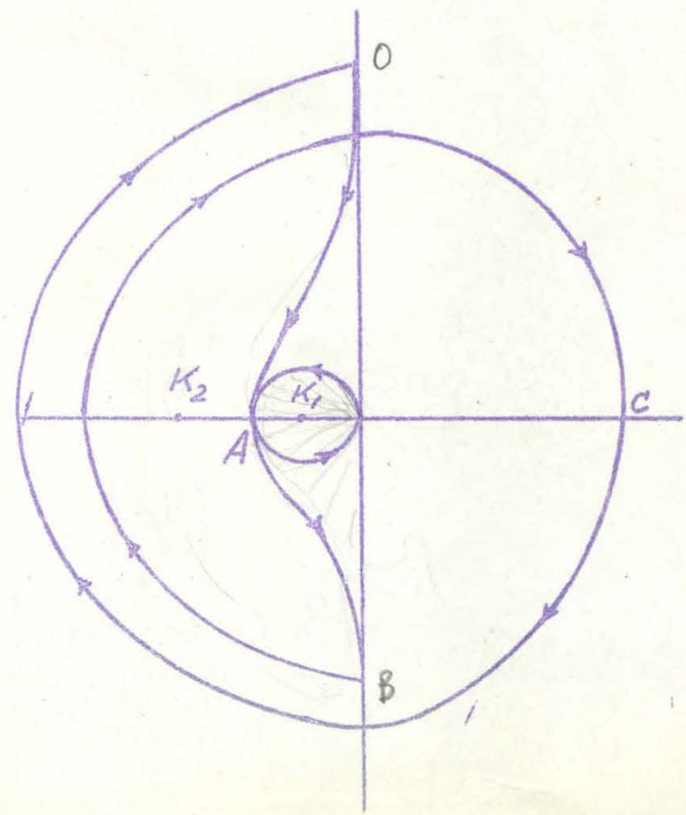
$$\frac{1}{G(i\omega)} = i\omega(1+i\omega T_1)(1+i\omega T_2) = i\omega M_1 e^{i\theta_1} M_2 e^{i\theta_2}$$

At $\omega = 0$, $1/G(i\omega) = 0i$

At $\omega \to +\infty$, $1/G(i\omega) = -i\infty$

The vector $M_1 e^{i\theta_1}$ and $M_2 e^{i\theta_2}$ which multiply $i\omega$ turn this vector progressively through 180°, i.e., from 90° to 180° to 270°. The end point points of the vector $1/G(i\omega)$ are represented by the curve OAB. As p traverses the portion of the imaginary axis from $\omega = 0$ to $\omega = -\infty$, the end points of the vector $1/G(i\omega)$ follow the curve OA'B', which is the mirror image of OAB in the real axis.





b) Along the infinite contour : $p = Re^{i\Phi}$ as $R \to \infty$.

$$\frac{1}{G(p)} \longrightarrow R^3 e^{i3\Phi}$$

As the vector p rotates 180° clockwise along the infinite circle from B to C to B', the vector $1/G(p)$ rotates 3 x 180° = 540° clockwise along the infinite circle.

Let us examine now whether the vector $1/G(p)$ encircles the point -K as p describes the contour C. It is obvious that $-K_1$ point involves no net rotation while $-K_2$ involves two clockwise revolutions. Therefore while the gain K_1 is stable, the gain K_2 is unstable. The point A signifies the cross-over point. This criterion for stability of feed back systems is called Nyquist's criterion; the diagram is called Nyquist's diagram for the $1/G(p)$. We do not, however, obtain any quantitative estimate from this diagram to indicate how unstable the system is. The root locus diagram gives the solution of this same problem including a quantitative measure of the negative damping coefficient as a function of the gain K.

In order to obtain a quantitative measure of the damping coefficient, positive or negative, it is in principle possible to carry out the conformal mapping not only of the line $p = i\omega$, but also of neighboring lines of constant damping $p = \lambda + i\omega$, keeping λ = constant. However this process is ordinarily too tedious for practical application.

R. M. Osborn* showed that for typical control systems an approximate measure of the damping coefficient ζ for the most critical root is given by

$$\zeta = \frac{1}{60}\frac{\alpha}{m} \qquad (8.14)$$

where α = "phase margin" in degrees $(180^\circ - \phi)$ at the cut off frequency where $|KG(i\omega)| = 1$ and m = slope of the curve of $\log_{10}|KG(i\omega)|$ vs

* "Criteria Relating Steady-State Response to Transcient Response of Closed Loop System" I.R.E. (1951)

$\log_{10}\omega$ at the point $|KG(i\omega)| = 1$. Thus if $\alpha = 30^\circ$, and $m = 1.7$, then $\zeta \cong \frac{1}{60} \times \frac{3.0}{1.7} = 0.3$.

Another word of caution is necessary when the function $G(p)$ has r zeros in the right half of the p- plane. Then the vector $1/G(p)$ will carry out r counterclockwise rotations as p carries out one clockwise rotation around the contour C due to these r zeros. If the $1/G(p)$ carries out r counterclockwise rotations around $-K$. That K is stable. But if $\frac{1}{G(p)}$ carries out $(r-s)$ counterclockwise rotations around $-K$, then this K is unstable and has s roots within the right half of the p- plane.

Bode diagrams is a plot of the magnitude and phase of $G(i\omega) = Me^{i\theta}$ in two separate diagrams: $\log_{10} M$ vs $\log_{10}\omega$ and θ vs $\log_{10}\omega$. According to (8.12), the point on these diagrams which belong to the root locus is the point for which $\theta = \pm 180^\circ$, the corresponding KM should be equal to unity. Therefore if at $\theta = \pm 180^\circ$, the corresponding M is M_{180°. then the critical gain K_c is $K_c = 1/M_{180^\circ}$. A desirable design condition in terms of the Bode diagram is frequently stated to be a "phase margin" of about 30° to 50° at "gain cross over", i.e., $\theta = 150^\circ$ to 130° at the frequency where $KM = 1$.

Comparison of Different Methods

In comparing the Bode or Nyquist presentation, with Evans method, we note that the Bode or Nyquist diagrams permit the direct use of experimental data. The Evans method requires the knowledge of the zeros and poles of the transfer function $G(p)$ and requires an approximate analytic representation of experimental data. This could be a disadvantage.

On the other hand, the Evans root-locus method presents directly a complete picture of the stability and transient response characteristics that are most important for aircraft control. In brief, it has the following merits:

a) The root-locus gives the roots of the closed-loop system directly and, by a simple calculation, the transient response.

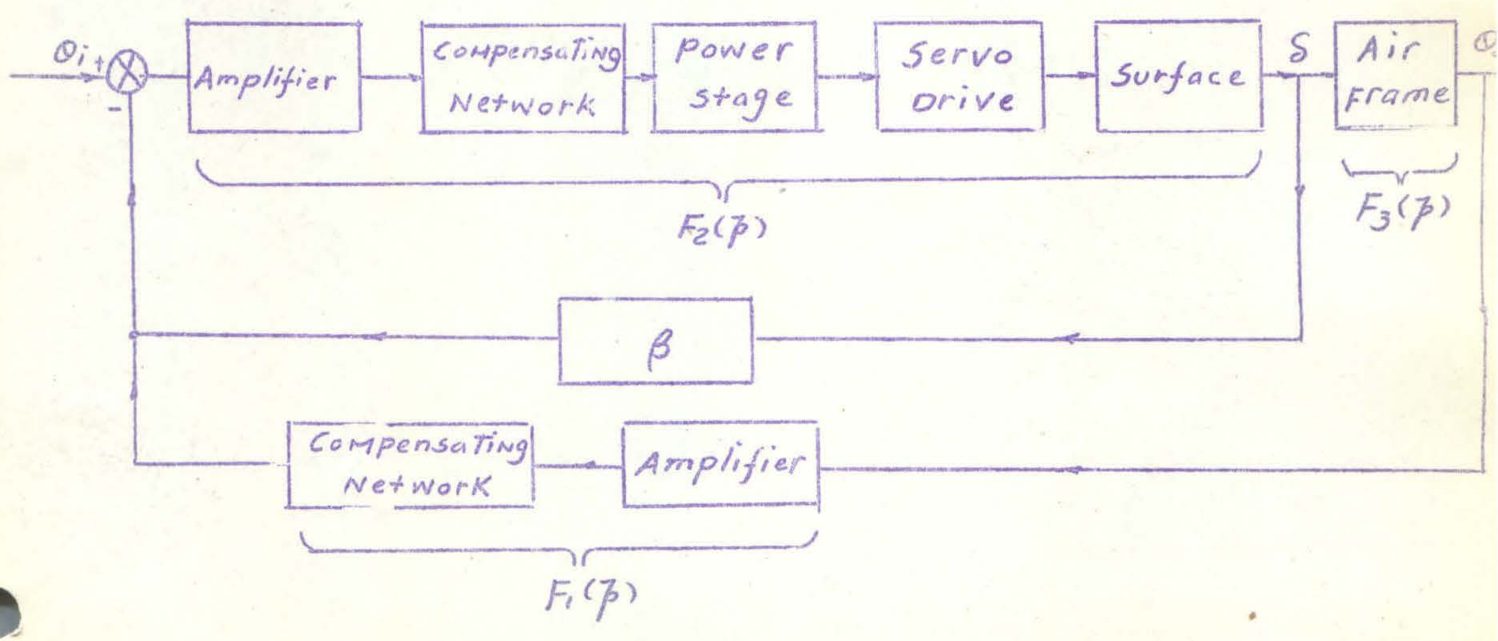
b) The degree of stability can be read from the root locus directly.

c) Complicated systems can be set up in such a fashion that the effect on transient response and stability of changing any parameter in the problem can easily be visualized.

d) There is no ambiguity or confusion concerning interpretation of the plots even for complex systems having any number of roots and poles in either the right or left half of the complex plane.

Multiple-Loop Networks

The systems discussed in the previous sections are really the simpler control systems. Ordinarily a control system is more complicated than these discussed. For instance a typical system for the control of an airplane about a single axis may be as follows in block diagram.



It is often advantageous, in spite of the increased system complexity engendered thereby, to add the feature of the control-surface position feedback or "follow-up". The advantages and disadvantages of such an addition were discussed by L. Becker.* If the inner loop is not closed, then we have the usual feedback control, and

$$\frac{\theta_o}{\theta_i} = \frac{F_2(p)\,F_3(p)}{1+F_1(p)\,F_2(p)\,F_3(p)} \tag{8.15}$$

If both loops are closed, then

$$\delta = F_2(p)\left[\theta_i - \beta\delta - F_1(p)\,\theta_o\right]$$

$$F_3(p)\,\delta = \theta_o$$

Therefore

$$\frac{\delta}{\theta_i} = \frac{F_2(p)}{1+F_1(p)\,F_2(p)\,F_3(p)+\beta F_2(p)} \tag{8.16}$$

and

$$\frac{\theta_o}{\theta_i} = \frac{F_2(p)\,F_3(p)}{1+F_1(p)\,F_2(p)\,F_3(p)+\beta F_2(p)} \tag{8.17}$$

The stability of the system then depends upon the roots of the polynomial in the denominator, or the zeros of $1+F_1(p)F_2(p)F_3(p)+\beta F_2(p)$. The

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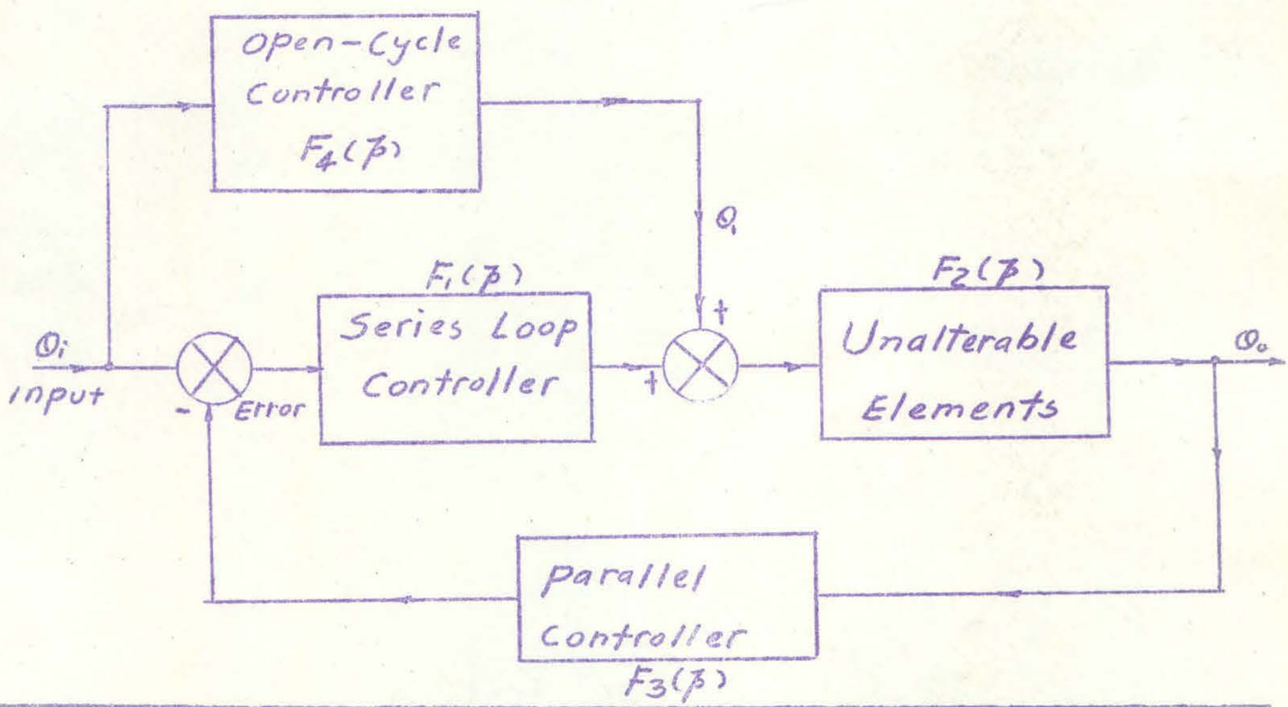
* L. Becker, "Control-Surface Position Feedback in Flight-Control Systems" Aeronautical Engineering Review Vol. 10 (1951), No. 9, September

graphical method, "Pin-Ball" method of Evans*, seems to be applicable to this case.

One of the difficulties of designing or "synthesizing" a good control system is the difficulty of having a large gain K for small steady state error, and having a stable system under dynamic conditions. This led to the idea of combining the closed-cycle control with open-cycle control, proposed by J. R. Moore.** An open cycle system attempts to produce a desired output by operations on the input without benefit of comparison of the output and input. Examples of such are found in any type of meter where the output depends upon known properties of the instrument. This idea can be extended with profit to some servo applications. Here, operations on input before the detector may be fed around the detector into the actuator amplifier as shown in the following figure.



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* Cf. W. Bollay, loc. cit. p. 613

** J. R. Moore, "Combination Open-cycle closed-cycle Systems" Proc. I.R.E. Vol. 39, pp. 1421-1432 (1951)

Then we have

$$F_4(p)\,\theta_i = \theta_1$$

$$\theta_o = F_2(p)\left[\theta_1 + F_1(p)\left\{\theta_i - F_3(p)\,\theta_o\right\}\right]$$

Solving for θ_o/θ_i , we get

$$\frac{\theta_o}{\theta_i} = \frac{F_2(p)F_4(p) + F_1(p)F_2(p)}{1 + F_1(p)F_2(p)F_3(p)} \tag{8.18}$$

Therefore if we are able to make

$$F_2(p)F_4(p) \sim 1, \quad \text{or } F_4(p) \sim \frac{1}{F_2(p)}$$

then the steady state error $(\theta_o/\theta_i) - 1$ will be very small even for very small gains in $F_1(p)$, $F_2(p)$ and $F_3(p)$. This is a fortunate situation, because while it is not difficult to "synthesis" a lead network $F_4(p)$ to completely compensate for $F_2(p)$, it will be extremely difficult to do that at a large gain. The combined system of open-cycle and closed-cycle control thus allows the design of the feedback loop primarily from stability and noise elimination considerations without jeopardizing "steady state" or synchronizing operations, since these are largely taken care of by the "open-cycle" portion of the system.

The most complicated system is that of automatic control and guidance of aircraft. Examples of that is given by J. B. Rea.* Here the problem of analysing such complicated system is a challange to all applied mechanicist.

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* J. B. Rea, "Analysis of Systems for Automatic Control of Aircraft" Aero Eng. Review Vol. 10, No. 11, Nov. (1951)

9. Automatic Control of Propulsion System

Most propulsion systems have two or more parameters that can be varied to produce a given thrust. During cruise, the important problem is to operate the system at the optimum specific fuel consumption expressed in pounds per mile. An open-cycle, closed-cycle system appears well adapted to schedule the operation at approximately the correct speed and altitude and to depend thereafter upon the direct measurement of fuel flow and distance in order to adjust the parameters for optimum performance. The adjustment of the controls must be carried out consistent with the limitations of the engine and the aircraft. The various adjustable parameters and the limiting factors are shown in the following table. All of these power plants suffer under certain conditions from instabilities. The first form may experience a pulsation associated with the compressor; the last form may experience instabilities associated with combustion. There is a good possibility, that automatic control systems may either prevent operation in these unstable regimes, or, might even damp out such oscillations by introducing a feedback link into the system.

In such control systems, the first task is to determine the transfer functions of the engine elements. Such elementary transfer functions are ratios of polynomials in p if the element has finite degrees of freedom. But if the mechanism has infinite many degrees of freedom, then strictly speaking more general transfer function is necessary to represent its characteristics. An example of this is the transfer function of rocket nozzle defined as the ratio of the practional increase in mass rate of flow and the fractional increase in chamber pressure. This transfer function is determined by the problem of non-steady flow of a continuum and thus represents the characteristics of infinitely many degrees of freedom. Whichever may be the case, the transfer function nevertheless define fully the characteristics of the mechanism so long as the characteristics are linear characteristics.

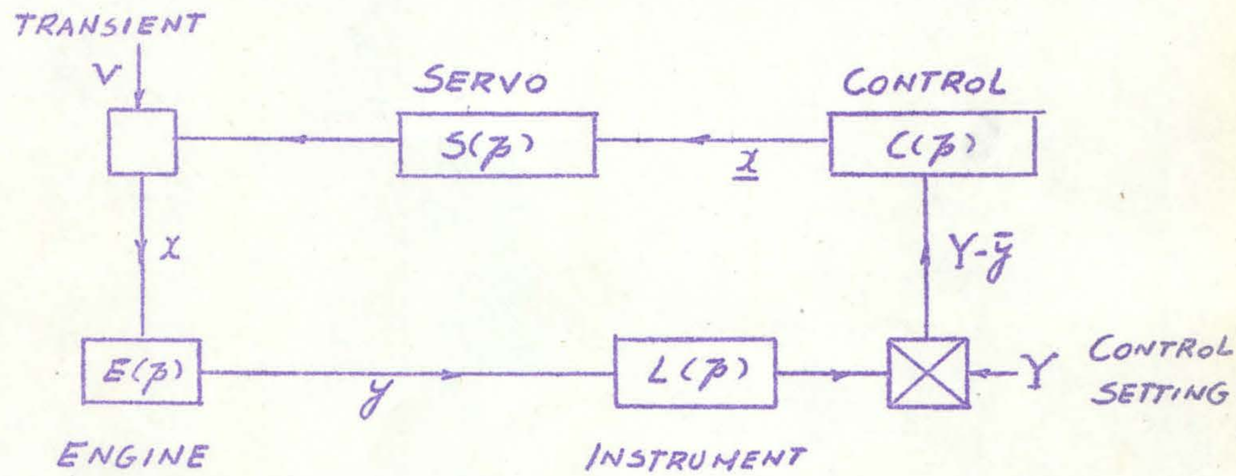
Table

Propulsion System	Adjustable Parameters	Principal Operating Limits
Piston Engine & Propeller	1. r.p.m. by propeller pitch	1. Maximum r.p.m.
	2. Manifold pressure by fuel flow	2. Maximum manifold pressure
Turbojet	1. r.p.m. by fuel flow	1. Maximum r.p.m.
	2. Turbine inlet temperature by exhaust nozzle area	2. Maximum turbine inlet temp.
		3. Pulsation limit
		4. Combustion blow-out limit
Propeller turbine	1. r.p.m. by propeller pitch	1. Maximum r.p.m.
	2. Turbine inlet temperature by fule flow	2. Maximum turbine inlet temp.
		3. Pulsation limit
		4. Combustion blow-out limit
Ramjet	1. Maximum temperature by fuel flow	1. Maximum temperature
		2. Pulsation limit
	2. Pressure recovery by exhaust nozzle area	3. Blowout limit
Rocket	1. Fuel flow	1. Maximum pressure
	2. Fuel ratios	2. Combustion instability

Actually even more restricted characterization is sufficient. The engine element is fully described by the frequency response characteristics. The frequency response is the transfer function with $p = i\omega$, $0 \leq \omega < \infty$, or the transfer function is defined along half of the imaginary axis in the complex p-plane. Then by the principle of analytic continuation, the transfer function is defined in the whole p-plane. The frequency response can be either calculated* or experimentally determined.** For reliable data to be used in system synthesis, the experimentally determined frequency response is preferred.

Boksenhom-Hood Theory of Control

The system considered here is essentially that shown in the following figure. This block diagram describes a typical engine and control system. The



general system configuration chosen is the one employing negative feedback in which error in the controlled variable is fed back into the control. The

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* E. W. Otto and B. L. Taylor, "Dynamics of Turbojet Engine Considered as a Quasi-Steady System" NACA TN 2091 (1950)

** H. Shames, S. C. Himmel, D. Blivas, "Frequency Response of Positive-Displacement Variable-Stroke Fuel Pump" NACA TN 2109 (1950).
B. L. Taylor, F. L. Oppenheimer, "Investigation of Frequency-Response Characteristics of Engine Speed for a Typical Turbo-propeller Engine" NACA TN 2184 (1950).

engine is represented as a box with a characteristic function E, the input of which is the engine-independent variable X and the output is the engine-dependent variable y. The characteristics of the instrument measuring the value of the controlled variable y are denoted as L for which the input is the variable y itself and the output is the measured value $\bar{y}$. The error is the difference between the setting Y and the measured value of the variable $\bar{y}$. The characteristics of the control are denoted as a function C for which the input is the error $Y - \bar{y}$ and the output is a signal $\underline{X}$ to the servo unit, which is the device that actually changes the engine-independent variable and completes the loop. The servo characteristic is denoted as the function S and may be considered as a part of the control or as a separate entity, as shown above.

In addition to the intentional disturbance made on the control setting, inadvertent transient disturbances may appear in any part of the system. One possible source of such transient disturbances has been included as an uncontrolled change in engine-independent variable. The variable X is thus the sum of a controlled part and an uncontrolled part V.

The characteristic functions used to describe the various units of the system are transfer functions. Thus for the system of above figure, the following equations can be written:

Engine characteristic:

$$y = E \cdot X \tag{9.1}$$

Control characteristic:

$$\underline{X} = C \cdot (Y - \bar{y}) \tag{9.2}$$

Instrument characteristic:

$$\bar{y} = L \cdot y \tag{9.3}$$

Servo and transient disturbance:

$$x = S \cdot \underline{x} + V \tag{9.4}$$

These equations can be combined for the effect of the disturbances Y and V on the controlled variable y as follows:

$$y = \frac{ESC}{ESCL+1} \cdot Y + \frac{E}{ESCL+1} V \tag{9.5}$$

These relations are more complex than the simple systems considered in the previous chapter, although the general principles and method of analysis are the same. However, for complex engine types, even this is not sufficient. The following very general treatment is due to A. S. Boksenbom and R. Hood.*

If the engine has more than one independent variable, there are additional degrees of freedom in control. The engine-characteristic function must be expanded to show how each dependent variable of interest is affected by all the engine inputs as follows:

$$\left.\begin{aligned} y_1 &= E_{11}x_1 + E_{12}x_2 + \cdots + E_{1n}x_n \\ y_2 &= E_{21}x_1 + E_{22}x_2 + \cdots + E_{2n}x_n \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \\ y_i &= E_{i1}x_1 + E_{i2}x_2 + \cdots + E_{in}x_n \end{aligned}\right\} \quad \begin{aligned} &y_j = E_{jk}x_k \\ &(j = 1, \cdots i) \\ &\;k = 1, \cdots n) \end{aligned} \tag{9.6}$$

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* A. S. Boksenbom and R. Hood: "General Algebraic Method Applied to Control Analysis of Complex Engine Types" NACA TR No. 980 (1950).

In this equation (9.6) each dependent variable y is expressed as a linear sum of effects due to each independent variable x. For n independent variables, there are n terms in each equation. There is an equation for each dependent variable of interest. For i dependent variables, where $i \leq n$, there are i equations.

The number of degrees of freedom of the entire system cannot be greater than the number of degrees of freedom of the engine alone. For continuously acting controls (excluding such controls as limiters), the total number of engine variables being controlled should not be greater than the number of engine-independent variables.

The equation (9.6) can be visualized in the matrix form. The engine functions are set up in a rectangular array where E_{jk} represents the element in the j_{th} row and the k_{th} column. The X inputs enter the matrix in the columns and the y outputs are attached to the rows. Each input multiplies all the elements in its column and each output is the linear sum of the resulting products in its row. For later use, a square matrix obtained by using only the first i columns is denoted by E^*.

$$
\begin{array}{cccc|cc l}
x_1 & x_2 & & x_i & & x_n & \\
\downarrow & \downarrow & & \downarrow & & \downarrow & \\
\hline
E_{11} & E_{12} & \cdots & E_{1i} & \cdots & E_{1n} & \rightarrow y_1 \\
E_{21} & E_{22} & \cdots & E_{2i} & \cdots & E_{2n} & \rightarrow y_2 \\
\cdots & \cdots & \cdots & \cdots & & & \\
\cdots & \cdots & \cdots & \cdots & & & \\
E_{i1} & E_{i2} & \cdots & E_{ii} & \cdots & E_{in} & \rightarrow y_i \\
\hline
\multicolumn{4}{c}{\underbrace{\hspace{5cm}}_{E^*}} & & &
\end{array}
$$

The control system is generalized to this extent: 1) Inasmuch as the engine has n independent variables, there are n degrees of freedom for the entire system. If only a limited number i of engine-dependent variables are to be controlled, the difference $n-i$ of the engine-independent variables can be controlled. 2) The control system employs negative feedback where only errors are applied to the control. 3) Each error is to affect every engine-independent variable.

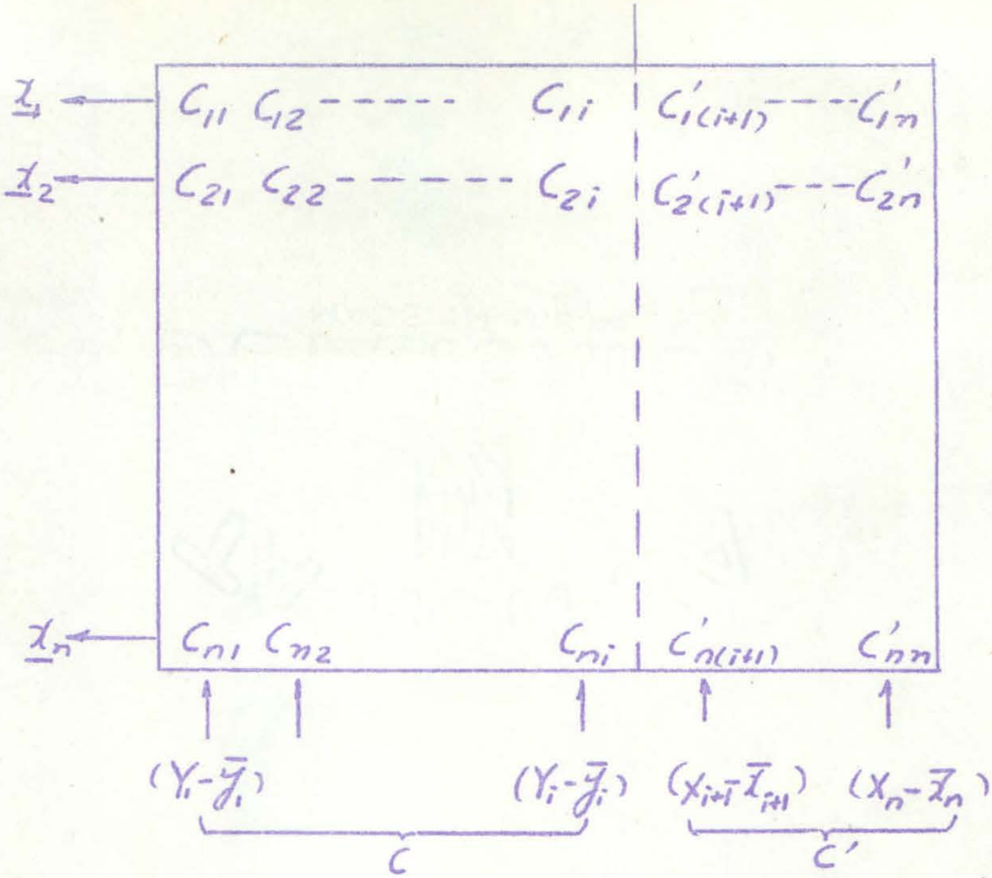
The control equations are written where each control output $\underline{X}$ is expressed as a linear sum of effects due to the errors in the controlled variables as follows:

$$\underline{X}_1 = C_{11}(Y_1-\bar{y}_1)+C_{12}(Y_2-\bar{y}_2)+----+C_{1i}(Y_i-\bar{y}_i)$$
$$+C'_{1(i+1)}(X_{i+1}-\bar{X}_{i+1})+---+C'_{1n}(X_n-\bar{X}_n)$$
$$-----------------------$$
$$\underline{X}_n = C_{n1}(Y_1-\bar{y}_1)+C_{n2}(Y_2-\bar{y}_2)+----+C_{ni}(Y_i-\bar{y}_i)$$
$$+C'_{n(i+1)}(X_{i+1}-\bar{X}_{i+1})+------+C'_{nn}(X_n-\bar{X}_n)$$

Or $$\underline{X}_k = C_{kv}(Y_v-\bar{y}_v)+C'_{ku}(X_u-\bar{X}_u) \qquad (9.7)$$

$$(k=1,2----n,\quad v=1---i,\quad u=i+1---n)$$

There are a total of n controlled quantities, n errors, and n terms in each equation. There are also n engine-independent variables and therefore n equations. Those independent variables to be controlled are given subscript numbers from $i+1$ to n and those controls to which the corresponding errors are applied are denoted by C'.



The characteristics of the insturments measuring the value of the controlled variable are L, i.e.,

$$\bar{y}_v = L_{vv}\, y_v \qquad v = 1,\ 2\ ---\ i \tag{9.8}$$

$$\bar{x}_\mu = L_{\mu\mu}\, x_\mu \qquad \mu = i+1,\ ---\ n \tag{9.9}$$

The general assumption is made that each engine-independent variable would require a servo, which would change the variable in accordance with its corresponding signal $\underline{x}$. A source of transient disturbance is included that may vary any of the engine-independent variables outside of the controlled variation in them. Thus, the expression for the engine-independent variables can be given by

$$x_k = S_{kk} z_k + V_k \quad (k = 1, 2 ---- n) \tag{9.10}$$

These equations can be visualized in the matrix form. The figure on the following page represents the general system of five independent variables. Three engine-dependent and two engine-independent variables are controlled. The entire system is enclosed except for the outside disturbances, which can be imposed on the system. These are the five settings of controlled variables Y and X and the five transient disturbances that have been included.

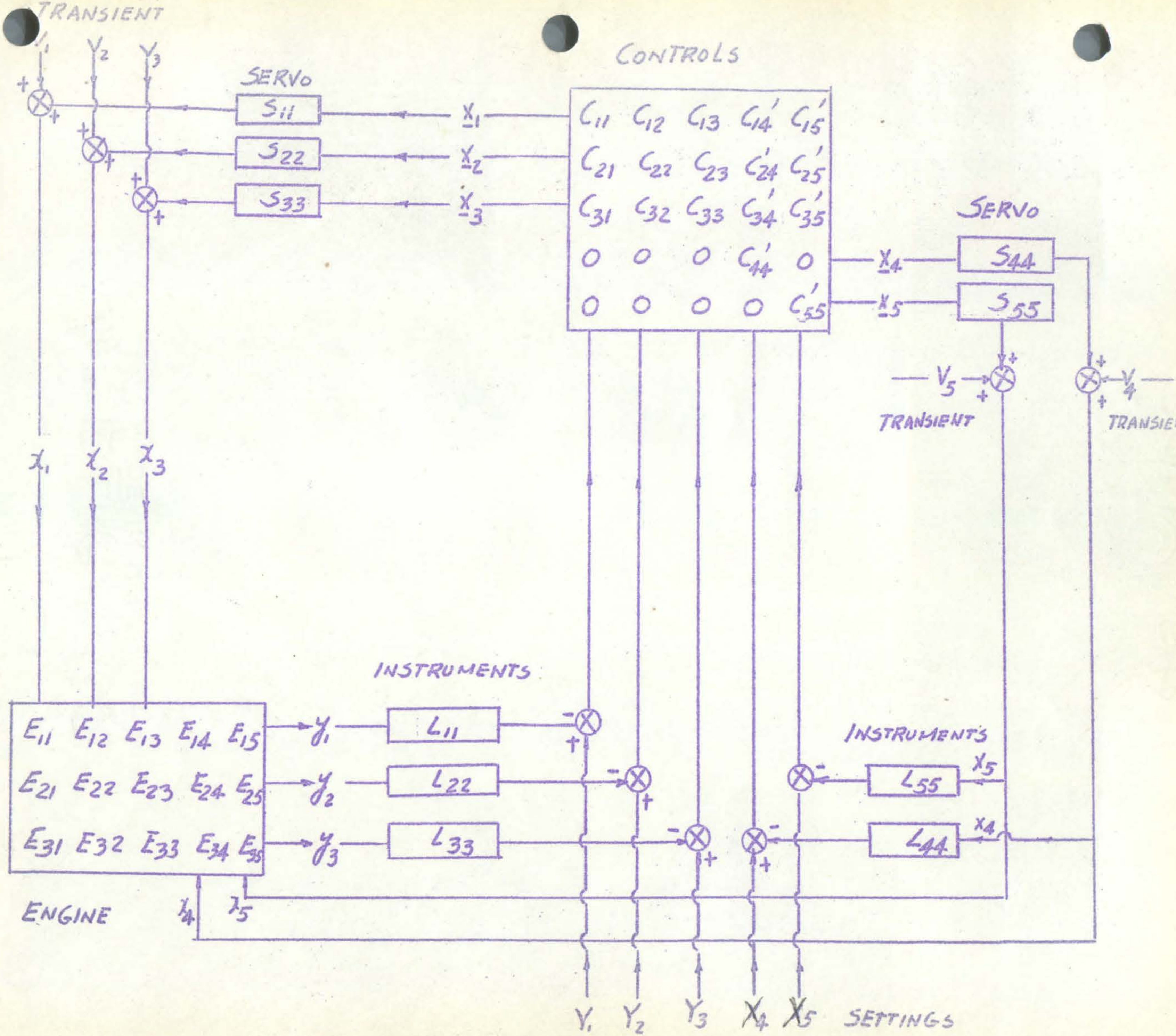
When the above equations are combined to give the effects on the controlled variables of the outside disturbances, the following equations are obtained:

$$y_j = E_{jk} S_{kk} C_{kv} (Y_v - L_{vv} y_v) + E_{jk} S_{kk} C'_{ku} (X_u - L_{uu} z_u) + E_{jk} V_k \tag{9.11}$$

and

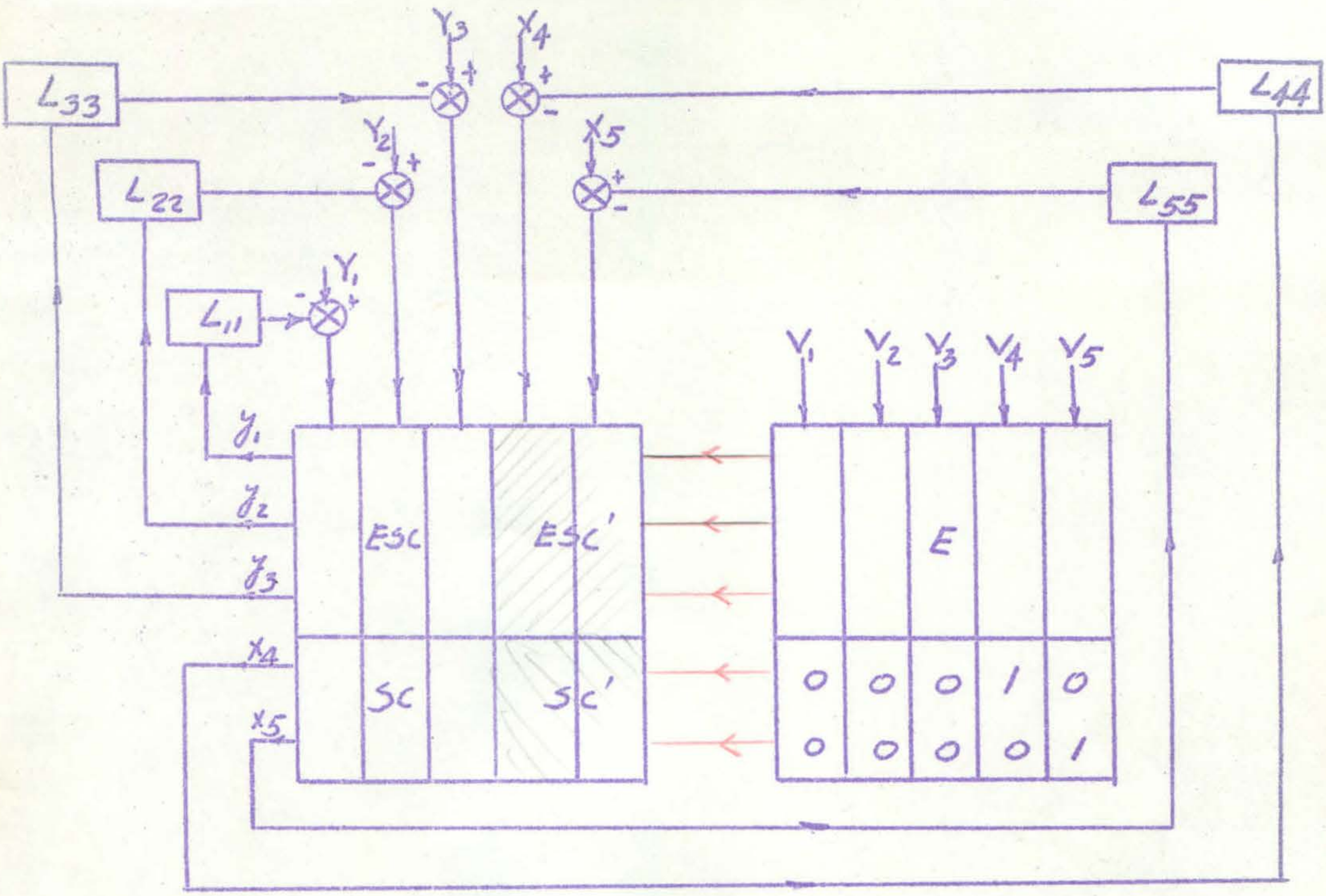
$$x_k = S_{kk} C_{kv} (Y_v - L_{vv} y_v) + S_{kk} C'_{ku} (X_u - L_{uu} z_u) + V_k \tag{9.12}$$

where $k = 1, 2 \cdots n, \quad j = 1, 2 \cdots i, \quad v = 1, 2, \cdots i, \quad u = i+1 \cdots n$



TRANSIENT
Y_1 Y_2 Y_3
SERVO
S_{11}
S_{22}
S_{33}
$\underline{X}_1$
$\underline{X}_2$
$\underline{X}_3$
CONTROLS
C_{11} C_{12} C_{13} C_{14}' C_{15}'
C_{21} C_{22} C_{23} C_{24}' C_{25}'
C_{31} C_{32} C_{33} C_{34}' C_{35}'
0 0 0 C_{44}' 0
0 0 0 0 C_{55}'
SERVO
$\underline{X}_4$
$\underline{X}_5$
S_{44}
S_{55}
V_5
V_4
TRANSIENT
TRANSIENT
X_1 X_2 X_3
INSTRUMENTS
E_{11} E_{12} E_{13} E_{14} E_{15}
E_{21} E_{22} E_{23} E_{24} E_{25}
E_{31} E_{32} E_{33} E_{34} E_{35}
y_1
y_2
y_3
L_{11}
L_{22}
L_{33}
INSTRUMENTS
L_{55}
L_{44}
X_5
X_4
ENGINE
X_4 X_5
Y_1 Y_2 Y_3 X_4 X_5 SETTINGS

A condensed form of matrix representation is shown in the following diagram:



A single-system matrix is used in which the inputs are the errors in controlled variables and the outputs are the controlled variables. The ESC matrix is a matrix in which the element in the j_{th} row and ν_{th} column is $E_{jk}S_{kk}C_{k\nu}$. Similarly, an ESC' matrix is a matrix in which the element in the j_{th} row and the μ_{th} column is $E_{jk}S_{kk}C'_{k\mu}$. An SC matrix is $S_{kk}C_{k\nu}$ and SC' is $S_{kk}C'_{k\mu}$. An additional matrix is included showing the effects of the transient disturbances.

Non-interaction Conditions

Controlling more than one variable generally introduces an interaction among the controlled variables. Making a new setting for one variable may cause, during the transient state, changes in other controlled variables. If these other variables are operating at or near a maximum point, this interaction may cause excessive values and possible damage to the engine. It would therefore be desirable that each new setting of a controlled variable affect only the variable being set, thus giving separate non-interacting control of all the variables being controlled.

Several types of non-interaction may be specified as follows:

(1) A specific control variable, such as $\mathcal{Y}_j$, is to be affected only by its own setting Y_j and is not to be affected by any other setting, which means that in the above figure the j th row of the system matrix contains only the diagonal element and the other element must be zero.

(2) Any specific setting, such as Y_t, is to affect only its corresponding controlled variable $\mathcal{Y}_t$ and not any of the other controlled variables. Thus the t th column of the system matrix contains only the diagonal element and the other elements must be zero.

(3) Every controlled variable is to be affected by its corresponding setting only, or every setting is to affect its corresponding controlled variable only. This complete non-interaction means that the system matrix is a diagonal matrix.

In general, the effect of the setting on the system is of primary interest and a condition such as (1) is of secondary interest. The interaction effects defined in terms of the effects of setting, such as (2) and (3) will thus be considered herein.

Eqs. (9.11) and (9.12) can be written as

$$y_j = \sum_{k=1}^{n} \sum_{\substack{v=1 \\ v \neq t}}^{i} E_{jk} S_{kk} C_{kv} (Y_v - L_{vv} y_v) + \sum_{k=1}^{n} \sum_{u=i+1}^{n} E_{jk} S_{kk} C'_{ku} (X_u - L_{uu} x_u) \tag{9.13}$$
$$+ \sum_{k=1}^{n} E_{jk} V_k + \sum_{k=1}^{n} E_{jk} S_{kk} C_{kt} (Y_t - L_{tt} y_t)$$

$$x_k = \sum_{\substack{v=1 \\ v \neq t}}^{i} S_{kk} C_{kv} (Y_v - L_{vv} y_v) + \sum_{u=i+1}^{n} S_{kk} C'_{ku} (X_u - L_{uu} x_u) + V_k \tag{9.14}$$
$$+ S_{kk} C_{kt} (Y_t - L_{tt} y_t)$$

If any setting Y_t is to affect only y_t , then any y_i is not affected by Y_t ($j \neq t$) and any controlled x_k ($k = i+1$ to n) is not affected by Y_t . If equation (9.13) is considered for any $j \neq t$, then only term that can be a function of Y_t is the last one. Because of the arbitrary nature of the instrument function, $Y_t \neq L_{tt} y_t$ so that

$$\sum_{k=1}^{n} E_{jk} S_{kk} C_{kt} = 0, \quad \text{if } j \neq t \tag{9.15}$$

If Eq. (9.14) is considered for $k = i+1$ to n, the only term that can be a function of Y_t is the last one; therefore

$$S_{kk} C_{kt} = 0$$

or

$$C_{kt} = 0 \tag{9.16}$$

where $\qquad k = i+1, ---- n$

Now by combining Eqs. (9.15) and (9.16),

$$\sum_{k=1}^{i} E_{jk} S_{kk} C_{kt} = \sum_{k=1}^{i} \delta_{jt} E_{tk} S_{kk} C_{kt} \tag{9.17}$$

where $\qquad j = 1, 2, ----- i$

and $\qquad \delta_{jt} = 0, \; j \neq t; \qquad \delta_{jt} = 1, \; j = t \tag{9.18}$

Eq. (9.17) is a system of i equations for the i unknowns $S_{kk}C_{kt}$. This system can be solved by using a property of matrix:

$$\sum_{p=1}^{i} E_{pk}|E^*_{pj}| = 0, \quad \text{When } k \neq j$$

and

$$\sum_{p=1}^{i} E_{pk}|E^*_{pj}| = |E^*| \quad \text{When } k = j$$

In these equations $|E^*_{pj}|$ indicates the cofactor of the E_{pj} element of E* matrix. Multiplying Eq. (9.17) by $|E^*_{pj}|$ and summing over p, we have

$$\sum_{k=1}^{i}\sum_{p=1}^{i} |E^*_{pj}|\delta_{pt}E_{tk}S_{kk}C_{kt} = \sum_{k=1}^{i}\sum_{p=1}^{i} E_{pk}|E^*_{pj}|S_{kk}C_{kt}$$
$$= |E^*|S_{jj}C_{jt}$$

Or

$$S_{jj}C_{jt} = |E^*_{tj}|\sum_{k=1}^{i} E_{tk}S_{kk}C_{kt}/|E^*| \qquad (9.19)$$

where $j = 1, 2, ---- i$

Because j is any number (1,2, ······ i), a similar expression for any other $S_{vv}C_{vt}$ is

$$S_{vv}C_{vt} = |E^*_{tv}|\sum_{k=1}^{i} E_{tk}S_{kk}C_{kt}/|E^*|$$

Thus

$$\frac{S_{jj}C_{jt}}{S_{vv}C_{vt}} = \frac{|E^*_{tj}|}{|E^*_{tv}|} \qquad (9.20)$$

This then gives the ratio of control and servo functions necessary for non-interaction. The result that only ratios are specified comes from the fact that the condition of (9.17) is really a set of i-1 equations for i unknowns C_{kt} ($t = 1, \cdots i$). We have shown that Eqs. (9.16) and (9.20) are necessary conditions for non-interaction. Actually they are also sufficient conditions for non-interactions of type (2), where Y_t affects only y_t.

Eqs. (9.11) and (9.12) can be also written as

$$y_j = \sum_{k=1}^{n}\sum_{v=1}^{i} E_{jk}S_{kk}C_{kv}(Y_v - L_{vv}y_v) + \sum_{k=1}^{n}\sum_{\substack{\mu=i+1\\ \mu\neq r}}^{n} E_{jk}S_{kk}C'_{k\mu}(X_\mu - L_{\mu\mu}z_\mu)$$
$$+ \sum_{k=1}^{n} E_{jk}V_k + \sum_{k=1}^{n} E_{jk}S_{kk}C'_{kr}(X_r - L_{rr}z_r) \qquad (9.21)$$

$$z_k = \sum_{v=1}^{i} S_{kk}C_{kv}(Y_v - L_{vv}y_v) + \sum_{\mu=1}^{n} S_{kk}C'_{k\mu}(X_\mu - L_{\mu\mu}z_\mu) + V_k$$
$$+ S_{kk}C'_{kr}(X_r - L_{rr}z_r) \qquad (9.22)$$

If any setting X_r is to affect only z_r, then any y_j is not affected by X_r and any controlled z_k ($k = i+1$ to n) is not affected by X_r for $k \neq r$. In equation (9.21) the only term that can be a function of X_r is the last one. Because of the arbitrary nature of the instrument function, $X_r \neq L_{rr}z_r$, so

$$\sum_{k=1}^{n} E_{jk}S_{kk}C'_{kr} = 0, \quad j = 1, \cdots i \qquad (9.23)$$

If equation (9.22) is considered for any k ($i+1, \cdots n$) where $k \neq r$, the only term that can be a function of X_r is the last one and therefore

$$S_{kk}C'_{kr} = 0$$

or

$$C'_{kr} = 0, \text{ for } k \neq r, \text{ and } k, r > i \qquad (9.24)$$

Now by combining the equations (9.23) and (9.24),

$$\sum_{k=1}^{i} E_{jk} S_{kk} c'_{kr} = -E_{jr} S_{rr} c'_{rr}$$

Thus

$$\sum_{p=1}^{i} \sum_{k=1}^{i} E_{pk} |\overset{*}{E}_{pj}| S_{kk} c'_{kr} = -S_{rr} c'_{rr} \sum_{p=1}^{i} |\overset{*}{E}_{pj}| E_{pr}$$

Therefore by using the basic matrix property again, we have

$$\frac{S_{jj} c'_{jr}}{S_{rr} c'_{rr}} = -\sum_{p=1}^{i} |\overset{*}{E}_{pj}| E_{pr} \Big/ |E^*| \tag{9.25}$$

where $\quad j = 1, 2, ---- i \quad (r > i)$

The necessary and sufficient conditions for non-interaction of type (2) for X_r are thus Eqs. (9.24) and (9.25).

If we wish to have complete non-interaction, where every setting is to affect its corresponding variable only, the necessary and sufficient conditions are specified by Eqs. (9.16), (9.20), (9.24) and (9.25).

Response Equations

The use of the operational functions to characterize each unit in the system allows algebraic representations of the relations among the variables of the system. With complete non-interaction specified, the system matrix becomes a diagonal matrix and each circuit acts as follows:

$$y_j = \sum_{k=1}^{i} E_{jk} S_{kk} c_{kj} (Y_j - L_{jj} y_j) + \sum_{k=1}^{n} E_{jk} V_k \tag{9.26}$$

and

$$x_r = S_{rr} c'_{rr} (X_r - L_{rr} x_r) + V_r \tag{9.27}$$

When these equations are solved for the controlled variables as a function of the settings and the transient disturbances, the following equations are obtained:

$$y_j = \frac{\sum_{k=1}^{i} E_{jk} S_{kk} C_{kj}}{\sum_{k=1}^{i} (E_{jk} S_{kk} C_{kj} L_{jj}) + 1} Y_j + \frac{\sum_{k=1}^{n} E_{jk} V_k}{\sum_{k=1}^{i} (E_{jk} S_{kk} C_{kj} L_{jj}) + 1} \tag{9.28}$$

$$x_r = \frac{S_{rr} C'_{rr}}{S_{rr} C'_{rr} L_{rr} + 1} X_r + \frac{V_r}{S_{rr} C'_{rr} L_{rr} + 1} \tag{9.29}$$

If the operational response function of y_j to Y_j is defined as R_{jj} and the operational response function of x_r to X_r as R'_{rr} , (9.28) and (9.29) becomes

$$y_i = R_{jj} Y_j - \sum_{k=1}^{n} (R_{jj} L_{jj} - 1) E_{jk} V_k \tag{9.30}$$

and $$x_r = R'_{rr} X_r - (R'_{rr} L_{rr} - 1) V_r \tag{9.31}$$

where

$$R_{jj} = \frac{\sum_{k=1}^{i} E_{jk} S_{kk} C_{kj}}{\sum_{k=1}^{i} (E_{jk} S_{kk} C_{kj} L_{jj}) + 1} \tag{9.32}$$

and $$R'_{rr} = \frac{S_{rr} C'_{rr}}{S_{rr} C'_{rr} L_{rr} + 1} \tag{9.33}$$

The response of any y_i to Y_j insofar as the control matrix is concerned depends only on the elements in the j th column of the control matrix. Inasmuch as for complete non-interaction only the ratios of the elements of that column of the control matrix are specified, the response can be independently set by choosing any one element of that column. If the conditions for non-interaction are used, R_{jj} is a function of any $C_{\nu j}$, i.e., by (9.20)

$$R_{jj} = \frac{\sum_{k=1}^{i} E_{jk} \left| E^{*}_{jk} \right| \frac{S_{vv} C_{vj}}{\left| E^{*}_{jv} \right|}}{L_{jj} \sum_{k=1}^{i} \left(E_{jk} \left| E^{*}_{jk} \right| \frac{S_{vv} C_{vj}}{\left| E^{*}_{jv} \right|} \right) + 1} = \frac{\left| E^{*} \right| S_{vv} C_{vj}}{\left| E^{*} \right| S_{vv} C_{vj} L_{jj} + \left| E^{*}_{jv} \right|} \tag{9.34}$$

(9.33) and (9.34) show the response of any controlled variable to its setting as determined by the control functions. The response of a controlled engine-dependent variable depends on the square engine matrix E^*.

An important result is obtained when equations similar to (9.28) and (9.29) are written for the errors in controlled quantities as follows:

$$Y_j - \bar{y}_j = Y_j - L_{jj} y_j = (1 - R_{jj} L_{jj})(Y_j - L_{jj} \sum_{k=1}^{n} E_{jk} V_k) \tag{9.35}$$

$$X_r - x_r = (1 - R'_{rr} L_{rr})(X_r - L_{rr} V_r) \tag{9.36}$$

The errors in controlled quantities respond in the same manner to both setting and transient disturbances and the control configuration chosen can act both as a control to setting disturbances and a regulator to transient disturbances, at least with the transient disturbances assumed.

Control Functions

Now suppose the response of the controlled variables are specified, i.e., R_{jj} and R'_{rr} are specified, we wish to determine the necessary control functions. Equations (9.33) and (9.34) can be solved for Cs'. Thus

$$S_{vv} C_{vj} = \frac{R_{jj}}{1 - R_{jj} L_{jj}} \frac{\left| E_{jv} \right|}{\left| E^{*} \right|} \tag{9.37}$$

and

$$S_{rr} C'_{rr} = \frac{R'_{rr}}{1 - R'_{rr} L_{rr}} \tag{9.38}$$

Equations (9.35) and (9.36) show that for the errors to be small, the numerators in (9.37) and (9.38) must be small. Therefore the gain of the control functions must be large. This is in accord with the conclusion of our previous study of simpler systems.

The algebraic methods used in dealing with operational functions can be further generalized in terms of equivalent relations when the engine, the control, and the other unit characteristics are expressed in other forms. For instance, if the frequency-response characteristics are graphically known from tests or analysis, all the results described here can be graphically performed in terms of frequency response characteristics in place of the operational functions.

Simple Examples.

If the turboprop engine, which has the two independent variables fuel flow and blade angle, is considered, the defining differential equations obtained by linearizing the general functional forms of engine torque as a function of speed N and temperature T, propeller torque as a function of speed N and blade angle β, and temperature T as a function of speed N and fuel flow w_f are

$$\left.\begin{aligned} N + \tau p N &= -a\beta + b w_f \\ T &= c w_f - eN \end{aligned}\right\} \tag{9.39}$$

Thus

$$N = \frac{-a}{1+\tau p}\beta + \frac{b}{1+\tau p} w_f \tag{9.40}$$

$$T = c w_f + \frac{ea}{1+\tau p}\beta - \frac{eb}{1+\tau p} w_f$$

Or $$T = \frac{ae}{1+\tau p}\beta + \frac{(c-be)+c\tau p}{1+\tau p}W_f \qquad (9.41)$$

Let us consider first the case of controlling speed N and fuel flow, W_f . Thus $i = 1$, $n = 2$. Here we see the reason why the general theory was developed for $n \neq i$. Of course we can also rearrange the system of Eqs. (9.40) and (9.41) such that the independent variables are T and β and the dependent variables are N and W_f . Then $n = i = 2$. But this additional work is not necessary if we use the general theory developed by Boksenbom and Hood. Thus

$$E_{11} = \frac{-a}{1+\tau p}; \qquad E_{12} = \frac{b}{1+\tau p}$$

$$x_1 = \beta, \qquad x_2 = W_f$$

$$y_1 = N, \qquad (y_2 = T)$$

The general control system is

$$\underline{\beta} = C_{11}(N_s - \bar{N}) + C_{12}'(W_{f,s} - \bar{W}_f)$$

$$\underline{W_f} = C_{21}(N_s - \bar{N}) + C_{22}'(W_{f,s} - \bar{W}_f)$$

where the subscript s denotes settings and $(\bar{\ })$ the measured values. For complete non-interaction then, according to (9.16)

$$C_{21} = 0$$

According to Eq. (9.25)

$$\frac{S_{11}C_{12}'}{S_{22}C_{22}'} = \frac{-|E_{11}^*|\,E_{12}}{|E^*|} = -\frac{E_{12}}{E_{11}} = \frac{b}{a}$$

where the determinant of a single-element matrix written as $|E^*| = |E^*_{11}|E_{11}$ allows the use of the equations employed here.

Problem 9.1

Show that for the non-interaction turboprop control indicated, (I) the control-system equations are

$$\underline{\beta} = C_{11}(N_s - \bar{N}) + \frac{b}{a}\frac{S_{22}}{S_{11}} C_{22}'(W_{f,s} - \bar{W}_f)$$

$$\underline{W}_f = C_{22}'(W_{f,s} - \bar{W}_f)$$

(II) the response of speed N and fuel flow W_f to their settings are

$$R_{11} = \frac{a S_{11} C_{11}}{a S_{11} C_{11} L_{11} - (1 + \tau p)}$$

$$R_{22}' = \frac{S_{22} C_{22}'}{S_{22} C_{22}' L_{22} + 1}$$

Problem 9.2

For the same turboprop engine considered but now control temperature T and speed N. Then n = i = 2. Deduce the conditions for complete non-interaction, and the control-system equations and response under complete non-interaction.

Answer:

(I) $$\frac{S_{11} C_{11}}{S_{22} C_{21}} = -\frac{1}{ae}\left[(c - be) + c\tau p\right]$$

$$\frac{S_{11} C_{12}}{S_{22} C_{22}} = \frac{b}{a}$$

(II) $$\underline{\beta} = C_{11}(N_s - \bar{N}) + \frac{b}{a}\frac{S_{22}}{S_{11}} C_{22}(T_s - \bar{T})$$

$$W_f = -\frac{ae}{(c - be) + c\tau p}\frac{S_{11}}{S_{22}} C_{11}(N_s - \bar{N}) + C_{22}(T_s - \bar{T})$$

(III) $$R_{11} = \frac{S_{22} S_{21}}{S_{22} S_{21} L_{11} + \frac{e}{c}}\;; \quad R_{22} = \frac{S_{22} C_{22}}{S_{22} C_{22} L_{22} + \frac{1}{c}}$$

Problem 9.3

Deduce Equations (9.26) and (9.27) from the given conditions for complete non-interaction.

Turbojet Engine with Tail-pipe Burning

As another example of the general theory, we shall consider the problem of controlling a turbojet engine with tail-pipe burning. The following analysis follows essentially the work of M. S. Feder and R. Hood.*

For close-to-equilibrium operating conditions, the torque Q available to accelerate the engine is a function of engine speed N, engine fuel flow F_e, effective nozzle exit area A, and the tail-pipe burner fuel rate F_t. For convenience, F_e and F_t are expressed in energy units per unit time. Thus

$$Q = \mathcal{F}(N, F_e, A, F_t) \tag{9.42}$$

If I is the rotary moment of inertia of the engine rotor, then

$$Q = I\dot{p}N = Ip\Delta N \tag{9.43}$$

where the Laplace transform is already applied. Thus in the following analysis, we do not make a different notation for the Laplace transform of a quantity and the quantity itself. No confusion will arise. ΔN is the increment of engine speed over the steady-state operating point. We can linearize (9.42) by considering only small deviations from the steady-state conditions. Thus

$$Ip\Delta N = \mathcal{F}_N \Delta N + \mathcal{F}_{F_e} \Delta F_e + \mathcal{F}_A \Delta A + \mathcal{F}_{F_t} \Delta F_t$$

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* M. S. Feder and R. Hood, "Analysis for Control Application of Dynamic Characteristics of Turbojet Engine with Tail-pipe Burning" NACA TN 2183, Sept. 1950.

where $\mathcal{F}_N$ for example is $\mathcal{F}_N = \frac{\partial \mathcal{F}}{\partial N}$

Thus

$$\left(-\frac{I p}{\mathcal{F}_N}+1\right)\frac{\Delta N}{N} = \left(-\frac{\mathcal{F}_{Fe}}{\mathcal{F}_N}\frac{F_e}{N}\right)\frac{\Delta F_e}{F_e} + \left(-\frac{\mathcal{F}_A}{\mathcal{F}_N}\frac{A}{N}\right)\frac{\Delta A}{A} + \left(-\frac{\mathcal{F}_{Ft}}{\mathcal{F}_N}\frac{F_t}{N}\right)\frac{\Delta F_t}{F_t}$$

We see that $-I/\mathcal{F}_N$ has the dimension of time and is the time constant of the system considered. Thus

$$\tau = -I/\mathcal{F}_N \tag{9.44}$$

The steady-state is represented by setting $p = 0$, then the factor

$$-\frac{\mathcal{F}_{Fe}}{\mathcal{F}_N}\frac{F_e}{N} = a_1 \tag{9.45}$$

represents the increment of $\Delta N/N$ per unit increment of $\Delta F_e/F_e$. Or it is the slope of a steady-state operating curve relating engine speed to engine fuel flow at constant A and F_t, considered at the initial steady-state operating condition. The other similar factors in the equation can be similarly interpacted. Thus by setting

$$-\frac{\mathcal{F}_A}{\mathcal{F}_N}\frac{A}{N} = a_2$$

and

$$-\frac{\mathcal{F}_{Ft}}{\mathcal{F}_N}\frac{F_t}{N} = a_3 \tag{9.46}$$

Then

$$\frac{\Delta N}{N} = \frac{a_1}{\tau p+1}\frac{\Delta F_e}{F_e} + \frac{a_2}{\tau p+1}\frac{\Delta A}{A} + \frac{a_3}{\tau p+1}\frac{\Delta F_t}{F_t} \tag{9.47}$$

By analysing the variation of turbine outlet temperature T_2, it is found that

$$\frac{\Delta T_2}{T_2} = \frac{\tau_1 p+1}{\tau p+1}b_1\frac{\Delta F_e}{F_e} + \frac{\tau_2 p+1}{\tau p+1}b_2\frac{\Delta A}{A} + \frac{\tau_3 p+1}{\tau p+1}b_3\frac{\Delta F_t}{F_t} \tag{9.48}$$

where b_1 is the slope of the steady-state relation between T_2 and F_e with A and F_t held constant, and similar interpretation for b_2 and b_3. The meaning of τ_1 can be obtained by making A and F_t constant, then the relation between N and T_2 is

$$\frac{\Delta N}{N} = \frac{1}{\tau_1 p + 1}\,\frac{a_1}{b_1}\,\frac{\Delta T_2}{T_2}$$

Thus τ_1 is the time constant for response of engine speed to changes in turbine outlet temperature T_2, with A and F_t held constant. Similarly τ_2 is the time constant for the response of engine speed to changes in turbine outlet temperature T_2, with F_e and F_t constant. And finally τ_3 is the time constant for the response of N to changes in T_2 with F_e and A held constant. All these time constants are however, not entirely independent, and related through the constants a_1, a_2, a_3, b_1, b_2, b_3 to τ In fact, by using the fact that there is a fixed relation, independent of time, between T_2, F_e, N, A, F_t, we can show that

$$\frac{b_1}{a_1}(\tau_1-\tau) = \frac{b_2}{a_2}(\tau_2-\tau) = \frac{b_3}{a_3}(\tau_3-\tau) \tag{9.49}$$

The previous relations are derived without any considerations on the engine thermodynamics. Therefore additional restraints on the constants can be obtained by using engine thermodynamics. This will be done in the following section.

Problem 9.4

Derive Eq. (9.49).

Engine Dynamics of Turbojet with Tail-pipe Burning

If the engine processes are considered quasi-static, thermodynamic relations that hold in the steady state can be extended to the transient state. Because unbalanced torque is a small difference between two large numbers, only those thermodynamic expressions that are precise and independent of engine efficiencies will be used. These expressions are the heat-balance equation and the continuity equation.

If H_e is heat added per lb. of air flow, H_2 and H_1 are the total enthalpy of air at compressor inlet and turbine outlet, then

$$H_e = \frac{F_e}{W_a} = \frac{QN}{W_a} + H_2 - H_1 \tag{9.50}$$

where W_a is the air weight rate. The term QN/W_a is the energy stored in accelerating the engine. By taking constant specific heats for differential changes in temperature, Eq. (9.50) gives after differentiation,

$$\frac{dF_e}{F_e} - \frac{dW_a}{W_a} = \frac{N\,dQ}{F_e} - \left(\frac{QN}{F_e W_a}\right) dW_a + \frac{H_2}{H_e}\frac{dT_2}{T_2}$$

the equilibrium condition is already introduced.

where the Ts are total temperatures at the designated stations. For deviations around steady-state conditions, the differentials in the equation are differential deviations from steady-state conditions and the remaining quantities are values of the variables at the initial steady-state condition. If deviations from steady state conditions are considered $Q = 0$, thus

$$\frac{\Delta F_e}{F_e} - \frac{\Delta W_a}{W_a} = \frac{N}{F_e}\Delta Q + \frac{H_2}{H_e}\frac{\Delta T_2}{T_2} \tag{9.51}$$

Now we shall introduce an approximation: In general, the term $\Delta W_a/W_a$ is a function of engine speed and compressor-inlet and-outlet conditions. If an axial-flow compressor is assumed, however, the air flow may be approximated as a function of engine speed alone; therefore

$$\frac{\Delta W_a}{W_a} = W_a' \frac{\Delta N}{N} \tag{9.52}$$

where W_a' is proportional to the slope of the steady-state relation between engine speed and air flow. By substituting Eqs. (9.43) and (9.52) into Eq. (9.51) we have

$$\frac{\Delta F_e}{F_e} - W_a'\frac{\Delta N}{N} = \frac{N}{F_e} I p \Delta N + \frac{H_2}{H_e}\frac{\Delta T_2}{T_2}$$

Or

$$\left(1 + \frac{IN^2 p}{F_e W_a'}\right)\frac{\Delta N}{N} = \frac{1}{W_a'}\left(\frac{\Delta F_e}{F_e} - \frac{H_2}{H_e}\frac{\Delta T_2}{T_2}\right)$$

By substituting Eq. (9.47) into this equation,

$$\left(1+\frac{IN^2 p}{F_e W_a'}\right)\left[\frac{a_1}{\tau p+1}\frac{\Delta F_e}{F_e}+\frac{a_2}{\tau p+1}\frac{\Delta A}{A}+\frac{a_3}{\tau p+1}\frac{\Delta F_t}{F_t}\right]=\frac{1}{W_a'}\left(\frac{\Delta F_e}{F_e}-\frac{H_2}{H_e}\frac{\Delta T_2}{T_2}\right)$$

Or

$$\frac{\Delta T_2}{T_2}=\frac{\frac{1}{1-a_1W_a'}\left(\tau-\frac{IN^2a_1}{F_e}\right)p+1}{\tau p+1}\;\frac{H_e}{H_2}(1-a_1W_a')\frac{\Delta F_e}{F_e}$$
$$-\frac{\frac{IN^2}{F_eW_a'}p+1}{\tau p+1}\;\frac{W_a'H_e}{H_2}a_2\frac{\Delta A}{A}-\frac{\frac{IN^2}{F_eW_a'}p+1}{\tau p+1}\;\frac{W_a'H_e}{H_2}a_3\frac{\Delta F_t}{F_t}$$

By comparing this equation with Equation (9.48), we have

$$\tau_2=\tau_3=\frac{IN^2}{F_e W_a'} \tag{9.53}$$

$$\tau_1=\frac{\tau-\tau_2 a_1 W_a'}{1-a_1W_a'} \tag{9.54}$$

and

$$\begin{aligned} b_1&=\frac{H_e}{H_2}(1-a_1W_a') \\ b_2&=-\frac{W_a'H_e}{H_2}a_2 \\ b_3&=-\frac{W_aH_e}{H_2}a_3 \end{aligned} \tag{9.55}$$

These equations show that the characteristics of the turbine outlet temperature T_2 response can be completely determined from the characteristics of the speed N response. This is, of course, to be expected, since the T_2 is completely determined by the energy relation, once W_a and F_e are specified. Feder and Hood gave another relation between a_2 and a_3. But the basis of this relation seems to be incorrect and therefore the relation is not valid.

Non-interacting Controls of Speed and Fuel Rates

Let us consider the problem of feed-back control when the settings are on N, F_e and F_t. In the Boksenbom-Hood theory, $i = 1$, $n = 3$. The engine equation is then Eq. (9.47), and

$$y_1 = \frac{\Delta N}{N}; \quad x_1 = \frac{\Delta A}{A}, \quad x_2 = \frac{\Delta F_e}{F_e}, \quad x_3 = \frac{\Delta F_t}{F_t}$$

and

$$E_{11} = \frac{a_2}{\tau p+1}, \quad E_{12} = \frac{a_1}{\tau p+1}, \quad E_{13} = \frac{a_3}{\tau p+1} \tag{9.56}$$

The general control system is

$$\left(\underline{\frac{\Delta N}{N}}\right) = C_{11}\left[\left(\frac{\Delta A}{A}\right)_s - \left(\overline{\frac{\Delta A}{A}}\right)\right] + C_{12}'\left[\left(\frac{\Delta F_e}{F_e}\right)_s - \left(\overline{\frac{\Delta F_e}{F_e}}\right)\right] + C_{13}'\left[\left(\frac{\Delta F_t}{F_t}\right)_s - \left(\overline{\frac{\Delta F_t}{F_t}}\right)\right]$$

$$\left(\underline{\frac{\Delta F_e}{F_e}}\right) = C_{21}\left[\left(\frac{\Delta A}{A}\right)_s - \left(\overline{\frac{\Delta A}{A}}\right)\right] + C_{22}'\left[\left(\frac{\Delta F_e}{F_e}\right)_s - \left(\overline{\frac{\Delta F_e}{F_e}}\right)\right] + C_{23}'\left[\left(\frac{\Delta F_t}{F_t}\right)_s - \left(\overline{\frac{\Delta F_t}{F_t}}\right)\right]$$

$$\left(\underline{\frac{\Delta F_t}{F_t}}\right) = C_{31}\left[\left(\frac{\Delta A}{A}\right)_s - \left(\overline{\frac{\Delta A}{A}}\right)\right] + C_{32}'\left[\left(\frac{\Delta F_e}{F_e}\right)_s - \left(\overline{\frac{\Delta F_e}{F_e}}\right)\right] + C_{33}'\left[\left(\frac{\Delta F_t}{F_t}\right)_s - \left(\overline{\frac{\Delta F_t}{F_t}}\right)\right]$$

In these equations, $(\underline{\ })$ means the output of the control matrix, $(\ )_s$ the setting and $(\bar{\ })$ the measured value from measuring instruments. The condition of non-interacting controls (9.16) then requires

$$C_{21} = C_{31} = 0 \tag{9.57}$$

The condition of (9.24) specifies

$$C_{32}' = C_{23}' = 0 \tag{9.58}$$

The condition of (9.25) then gives

$$\frac{S_{11} C_{12}'}{S_{22} \cdot C_{22}'} = -\frac{E_{12}}{E_{11}} = -\frac{a_1}{a_2} \tag{9.59}$$

and
$$\frac{S_{11} G_3'}{S_{33} C_{33}'} = -\frac{E_{13}}{E_{11}} = -\frac{a_3}{a_2} \tag{9.60}$$

The response is then

$$\left(\frac{\Delta N}{N}\right) = \frac{E_{11} S_{11} C_{11}}{E_{11} S_{11} C_{11} L_{11} + 1} \left(\frac{\Delta N}{N}\right)_s + \frac{E_{11} V_1 + E_{12} V_2 + E_{13} V_3}{E_{11} S_{11} C_{11} L_{11} + 1}$$

$$\left(\frac{\Delta Fe}{Fe}\right) = \frac{S_{22} C_{22}'}{S_{22} C_{22}' L_{22} + 1} \left(\frac{\Delta Fe}{Fe}\right)_s + \frac{V_2}{S_{22} C_{22}' L_{22} + 1}$$

$$\left(\frac{\Delta Ft}{Ft}\right) = \frac{S_{33} C_{33}'}{S_{33} C_{33}' L_{33} + 1} \left(\frac{\Delta Ft}{Ft}\right)_s + \frac{V_3}{S_{33} C_{33}' L_{33} + 1}$$

Therefore with the non-interaction conditions of Eqs. (9.57) to (9.60) satisfied. The problem of response is very much the same as the simple feed-back systems discussed in the previous chapter. The primary aim is to design C_{11}, C_{22}' and C_{33}' such that the gain is large and the system transfer functions are stable.

It is often required that turbine inlet temperature or the combustion outlet temperature be limited to safe value. Strictly speaking, this is a limit control and is a non-linear effect not easily analysed. However an approximate solution to this problem might be the control of this temperature, instead of the engine fuel rate. If the control is stable, then we know the steady state temperature will be within bond set by the control. Then the transfer function can be so designed as to give a asymptotic approach to the steady state temperature to ensure no overshoot. Or, perhaps, more practical, the transfer function can be so synthetized as to have a very short time characteristics. Then even with temperature overshoot, the effect is harmless. For a discussion on such aspects of engine control, one should consult a paper by F. C. Mock,* ~~and a report by A. R. Boksenbom and R. Hood.**~~

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* F. C. Mock, "Turbojet and Turboprop Engine Controls" Quarterly Transcation of SAE, Vol. 5, No.3, (July), pp. 377-392 (1951).

~~** A. R. Boksenbom and R. Hood, "Automatic Control Systems Satisfying Certain General Criterions on Transient Behavior" NACA TN 2378 (1951).~~

10. Servomechanisms with Alternating Current Motors and Oscillating Control Servomechanisms

In this chapter and the two following chapters, we shall extend the concepts and methods developed in the previous chapters on simple servomechanisms to systems which are basically more complicated but nevertheless could be treated approximately by the same technique. Therefore, they demonstrate the power of these basic principles of servomechanisms. The contents of this and the next chapter follow closely the treatment of L. A. MacColl.*

Alternating Current Systems

So far, whenever we have been considering a servomechanism containing an electrical motor, we have assumed implicitly that the motor is a direct current motor. Similarly, if the "motor" in a servomechanism is some mechanical system, we have been assuming that it is a "zero-frequency" device. In pratice, however, it may well be desirable to use alternating current motors. It is clear that the use of such motors necessitates the reconsideration of some parts of the forgoing discussion.

Consider a servomechanism as sketched in the following figure: In this system all of the currents and voltages appearing in the amplifier, motor and potentometer are modulated sinusoids. The basic alternating current is generated by the oscillator. When a certain condition, which will be discussed presently, is satisfied, much of the earlier theory is applicable to this system.

Let us consider for a moment the general steady-state theory of invariable linear systems subjected to signals which are modulated sinusoids. Here the expression "steady state" refers to the fact that the modulating signals are assumed to be sinusoidal functions of time.

If the unmodulated "carrier" is $\cos \omega_0 t$, the phase angle is here neglected without loss of generality. Then it is obviously legitimate to

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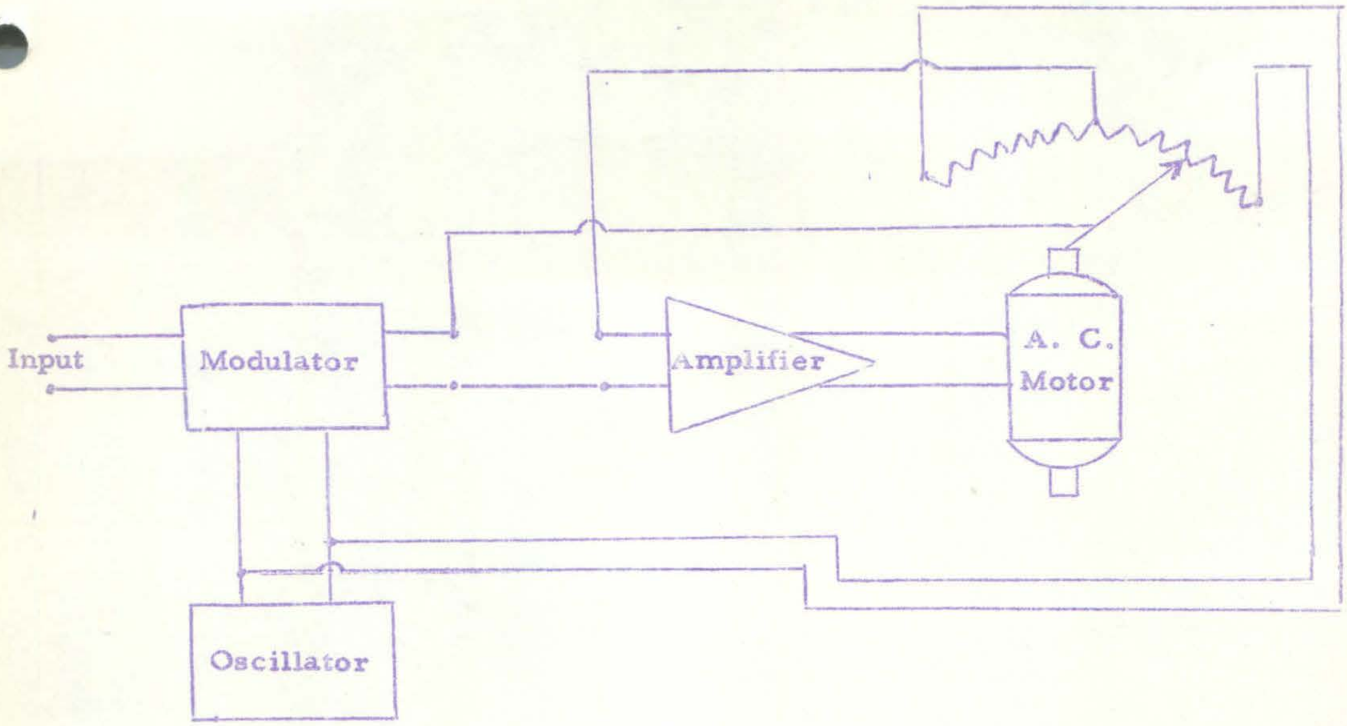
* "Fundamental Theory of Servomechanisms" D. Van Nostrand Co., New York, (1945).



take the modulating signal in complex form, $e^{i\omega t}$. Then the modulated carrier is

$$\frac{1}{2}\left[e^{i(\omega+\omega_0)t} + e^{i(\omega-\omega_0)t}\right] \tag{10.1}$$

and the steady-state response of a system having the transmission ratio $KG(i\omega) = F(i\omega)$ is

$$\frac{1}{2}\left[F(i\omega+i\omega_0)e^{i\omega_0 t} + F(i\omega-i\omega_0)e^{-i\omega_0 t}\right]e^{i\omega t} \tag{10.2}$$

For any real physical system, the function $F(i\omega)$ is generally ratio of polynomials with real coefficients. Then

$$F(-i\omega) = \overline{F(i\omega)} \tag{10.3}$$

where the bar over the symbol indicates the conjugate complex value. Therefore, we can write (10. 2) as

$$\frac{1}{2}\left[F^{*}(i\omega)e^{i\omega_0 t}+\overline{F^{*}(-i\omega)}\,e^{-i\omega_0 t}\right]e^{i\omega t} \tag{10.4}$$

where

$$F^{*}(i\omega)=F(i\omega+i\omega_0) \tag{10.5}$$

Now we suppose that the system is such that we have the relation

$$F(i\omega_0+i\omega)=\overline{F(i\omega_0-i\omega)} \tag{10.6}$$

Then the expression of (10. 4) can be written in the form

$$F^{*}(i\omega)e^{i\omega t}\cos\omega_0 t$$

This result shows that when the condition of (10. 6) is satisfied, the amplitude, $F^{*}(i\omega)e^{i\omega t}$, of the response of the system to the modulated carrier (10. 1) is the same as the response of a system having the transmission ratio $F^{*}(i\omega)$ to the input signal $e^{i\omega t}$. This statement can be immediately generalized to more general input functions by the principle of superposition: If (10. 6) is satisfied, at least approximately, throughout a range of values of ω which includes the more important parts of the Fourier spectrum of a modulating input signal $y_i(t)$, the amplitude of the resulting modulated output signal is at least approximately equal to the response of a system having the transmission ratio $F^{*}(i\omega)$ to the input signal $y_i(t)$.

Translation of $F(i\omega)$ to a Higher Frequency

If we leave out of account certain trivial systems, e. g., pure resistances, it follows from (10. 3)

$$F(i\omega_0 + i\omega) = \overline{F(-i\omega_0 - i\omega)}$$

This is different from the condition of (10.6). Therefore (10.6) cannot be satisfied exactly for all real values of ω. Or, if we alter our point of view slightly, we can say that the transmission ratios $F(i\omega)$ and $F^*(i\omega)$ of two physical systems cannot rigorously satisfy the relation of (10.5) for all real values of ω.

Nevertheless, it is entirely possible, and indeed quite common, for the transmission ratios $F^*(i\omega)$ and $F(i\omega)$ of two physical systems to satisfy the relation (10.5) approximately over a range of values of ω which is large enough to include the more important parts of the Fourier spectra of the particular signals we are concerned with. We can see this briefly as follows:

Consider the impedance Z of an inductance L and a capacitance C in series,

$$\begin{aligned} Z &= Li\omega + \frac{1}{Ci\omega} \\ &= Li\omega\left(1 - \frac{1}{LC\omega^2}\right) \end{aligned}$$

If we make L and C of such magnitude that

$$\omega_0^2 = 1/LC \tag{10.7}$$

then

$$Z = Li\omega\left(1 - \frac{\omega_0^2}{\omega^2}\right) = Li(\omega - \omega_0)\left(1 + \frac{\omega_0}{\omega}\right)$$

For $\omega - \omega_0$ small or ω near ω_0,

$$Z \cong 2Li(\omega - \omega_0)$$

That is, at the frequency $\omega_0 + \omega$ the impedance of the series combination of L and C, satisfying condition (10.7), is approximately equal to the impedance of an inductance $2L$ at the frequency ω.

Similarly, consider the impedance Z of an inductance L and a capacitance C in parallel

$$\frac{1}{Z} = \frac{1}{Li\omega} + Ci\omega$$
$$\cong 2Ci(\omega-\omega_0)$$

if condition of (10.7) is satisfied. Therefore at the frequency $\omega+\omega_0$ the impedance of the parallel combination of L and C, satisfying (10.7), is approximately equal to the impedance of a capacitance 2C at the frequency ω .

Thus, starting from a physical system with a transmission ratio $F^*(i\omega)$, we can, by replacing any inductance L by a series combination of inductance $\frac{1}{2}$ L and capacity $C = 2/L\omega_0^2$ and by replacing any capacity C by a parallel combination of capacity $\frac{1}{2}$ C and inductance $L = 2/C\omega_0^2$, obtain a physical system having a transmission ratio $F(i\omega)$ such that the relation (10.6) is satisfied approximately for small values of ω . Then the results stated above, concerning the relations between the signals in the systems, hold for these related systems.

After this excursion to general carrier transmission theory, let us return to the consideration of the servomechanism.

Let $\omega_0/2\pi$ denote the frequency of the current supplied by the oscillator. It is clear that all of the currents and voltages in the system are modulations of the carrier wave $\cos\omega_0 t$. Hence it immediately follows from the above that, in order to design the amplifier for the system with alternating current, we need only design a suitable amplifier for a system with direct current by methods described previously; and then translate the characteristics of the amplifier upward on the ω - scale by the amount ω_0 , in accordance with the procedure sketched above.

As we have indicated, the forgoing arguments involve a considerable number of approximations of one kind or another. An entirely complete discussion of servomechanisms with alternating current would necessitate an examination of the effects of all these approximations. We shall not go into this investigation, because it would be involved and tedious, and because it does not appear to be a very urgent matter as far as servomechanism art is concerned.

Oscillating Control Servomechanism

We shall now consider another class of systems, which we call oscillating control servomechanisms. These resemble servomechanisms with alternating current motors, in that in both cases the signals are caused to modulate a periodic oscillation. However, in the case of oscillating control servomechanisms the modulation employed is not the ordinary amplitude modulation. In order to introduce the concept of an oscillating control servomechanism intelligibly, we must first give a little preparatory discussion.

One very primitive, but very common, kind of servomechanism can be described as follows. Suppose that to the system we were to add a circuit containing a relay, designed to function so that no voltage would be applied to the output terminals unless the absolute value of the error $\mathcal{E}(t)$ exceed a certain threshold, and so that when $|\mathcal{E}|$ did exceed the threshold the output would be the full electromotive force E of a source applied with such a polarity as to tend to reduce the absolute value of the error. We would then have an example of what we call an on-off servomechanism.

On-off servomechanisms have the great advantage that comparatively simple systems of this kind can be made to handle large amounts of power. This is often difficult to achieve with servomechanisms of other types. On the other hand, on-off servomechanisms are definitely non-linear systems, and their performances tend to be inferior to those of the systems we have considered previously. Briefly, an oscillating control servomechanism is a modification of an on-off servomechanism, which enables us to secure the advantage of linearity without sacrificing the advantage of the large power carrying capacity.*

Before proceeding to the treatment of oscillating control servomechanisms proper, we shall present a general theoretical result, upon which the theory of all such systems is based.

Let us consider a device having the following property: According as the input signal, $y_i(t)$, is positive or negative, the output signal,

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* See also I. Flügge-Lotz: "Über Bewegungen eines Schwingers unter dem Einfluss von Schwarz-Weiss-Regelungen" Z.a.M.M. Vol. 25/27, pp. 97-113 (1947).

$y_o(t)$, is + A or - A, where A is a fixed constant. We may think of such a device as an ideal limiter. In fact, such a system is an on-off system with zero threshold.

Suppose that the input signal to the limiter is

$$y_i(t) = E_0 \sin \omega_0 t + k E_0 \sin \omega t \tag{10.8}$$

where E_0, k, ω_0 and ω are constants. In connection with oscillating control servomechanisms, the term $E_0 \sin \omega_0 t$ will be a persistent oscillation in the system, and $kE_0 \sin \omega t$ will be an applied signal or modulating signal. We shall calculate the corresponding output $y_o(t)$ presently.

Response of a Limiter

The output $y_o(t)$ of the limiter in response to the input (10.8) can be written in the form

$$y_o(t) = \sum_{m=0}^{\infty}{}' \sum_{n=-\infty}^{\infty}{}' a_{mn} \sin(m\omega_0 + n\omega)t \tag{10.9}$$

where the a's are independent of t. When $m = 0$, the inner summation is to be extended only over positive values of n. For our purposes the only coefficients that are of any immediate interest are a_{10} and a_{01}; for in the case of an oscillating control servomechanism, operating under normal conditions, the other coefficients either are negligibly small, or they correspond to oscillations which are suppressed by suitable filtering.

When $k = 0$, the coefficient a_{10} can be easily calculated as

$$a_{10} = A \int_0^{\frac{\pi}{\omega_0}} \sin \omega_0 t \cdot dt \Bigg/ \int_0^{\frac{\pi}{\omega_0}} \sin^2 \omega_0 t \cdot dt$$

Or

$$a_{10} = \frac{2\omega_0}{\pi} A \int_0^{\frac{\pi}{\omega_0}} \sin \omega_0 t \cdot dt = \frac{4A}{\pi} \tag{10.10}$$

When $k \neq 0$, the output $y_o(t)$ can be graphically obtained as follows:

It is seen that if $|k| \ll 1$, the corrections to the output from the case zero k is a series of narrow rectangles of heights 2A and duration $\frac{k}{\omega_0} \sin \omega t_n$ where $t_n = n\pi/\omega_0$. Therefore

$$a_{01} \int_0^{N\pi/\omega_0} \sin^2 \omega t \cdot dt = 2A \frac{k}{\omega_0} \sum_{n=0}^{N} \sin^2 \omega t_n$$

But

$$\int_0^{N\pi/\omega_0} \sin^2 \omega_0 t \cdot dt = \frac{1}{2} \int_0^{N\pi/\omega_0} [1 - \cos 2\omega t] \, dt$$

$$= \frac{1}{2} \frac{N\pi}{\omega_0} - \frac{1}{4\omega} \sin 2N\pi \frac{\omega}{\omega_0}$$

and

$$\sum_{n=0}^{N} \sin^2 \omega t_n = \sum_{n=1}^{N} \sin^2 n \left(\pi \frac{\omega}{\omega_0}\right)$$

$$= \frac{1}{2} \sum_{n=1}^{N} \left\{1 - \cos\left(2n\pi \frac{\omega}{\omega_0}\right)\right\} = \frac{N}{2} - \frac{1}{2} \sum_{n=1}^{N} \cos\left(2n\pi \frac{\omega}{\omega_0}\right)$$

The sum remains finite as we increase N indefinitely. Therefore by making N large, we have

$$a_{01} = 2 \frac{Ak}{\pi} \tag{10.11}$$

Eqs. (10.10) and (10.11) give the two important coefficients a_{10} and a_{01} for small k. For general values of k, these coefficients were computed by Bennett and Kalb.* When $0 < k < 1$,

$$\left.\begin{aligned} a_{10} &= \frac{8A}{\pi^2} E(k) \\ a_{01} &= \frac{8A}{\pi^2 k} \left[E(k) - (1-k^2) K(k)\right] \end{aligned}\right\} \tag{10.12}$$

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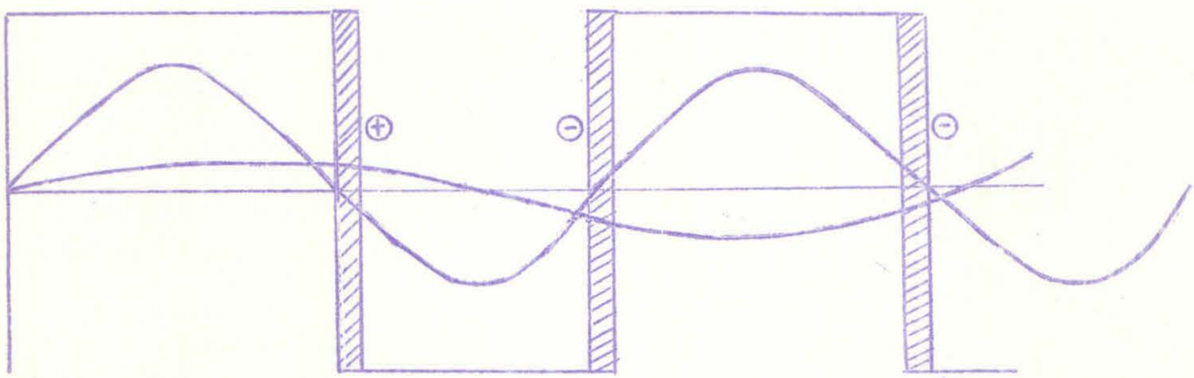
* Bell System Technical Journal, Vol. 14, pp. 322-359 (1935).

where K(k) and E(k) denote the complete elliptic integrals of the first and the second kinds, respectively. For k small, the elliptic integrals can be expanded, then

$$\left.\begin{aligned} a_{10} &= \frac{4A}{\pi}\left[1 - \frac{k^2}{4} - \cdots\right] \\ a_{01} &= \frac{2Ak}{\pi}\left[1 + \frac{k^2}{8} + \cdots\right] \end{aligned}\right\} \qquad (10.13)$$



Eq. (10.13) shows that our simple computation to be correct, within the accuracy of analysis. It, however, also shows that the simple results of Eqs. (10.10) and (10.11) are accurate enough for even moderate values of k. Therefore the ratio of the component of frequency ω in the output to the component of the same frequency in the input is approximately equal to

$$F(i\omega) = 2A/(\pi E_0) \qquad (10.14)$$

By Eq. (10.10), when k is small the amplitude of the component of frequency ω_0 in the output is approximately a constant, determined entirely by the properties of the limiter. Also, the ratio of the component of frequency ω_0 in the output to the component of the same frequency in the input is $4A/(\pi E_0)$. Thus the amplification of the limiter for the component of frequency ω_0 is 6 lb. greater than the amplification for the component of frequency ω.

Now it is easily seen that the preceding considerations can be extended to the case in which we have, instead of the signal $kE_0 \sin \omega t$, a small signal $f(t)$ of arbitrary form. The essential result can be stated as follows:

If we ignore higher order modulation terms (on the ground that they are negligibly small, or that they are to be suppressed ultimately by suitable filtering), the limiter with the input

$$E_0 \sin \omega_0 t + f(t)$$

where $f(t)$ is small compared with E_0, behaves, as far as the transmission of the signal $f(t)$ is concerned, substantially as a linear system having the transmission ratio given by Eq. (10. 14).

Limiter Servomechanisms with Built-in Oscillation

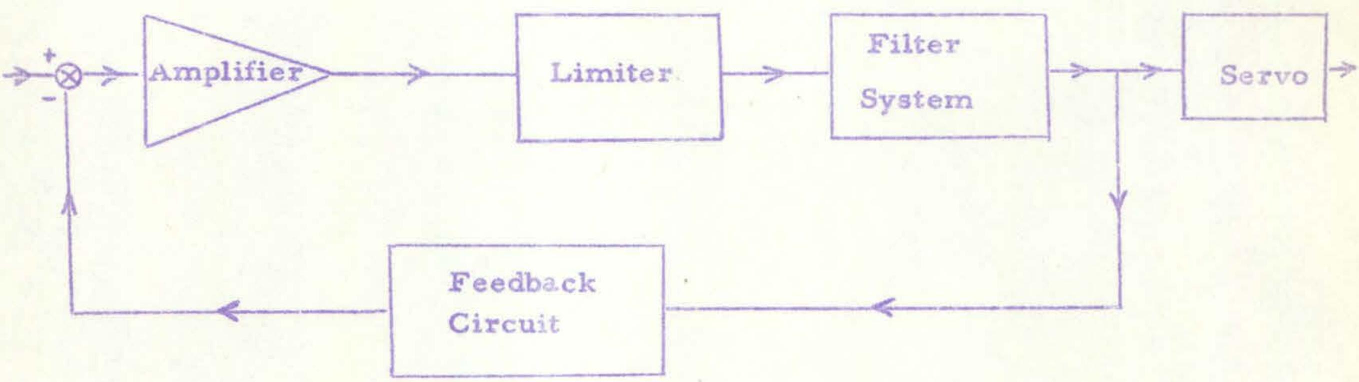
We are now ready to discuss the oscillating control servomechanism. We have already seen that the only effect of the persistent oscillation, as far as the transmission of signals is concerned, is to make the limiter into an effectively linear element, with a positive real transmission ratio. Hence, we might have considered the limiter to be such an element from the beginning, avoided all explicit mention of the oscillation $E_0 \sin \omega_0 t$, and dealt with the system entirely by means of the concepts and methods given in the earlier chapters. This is, in fact, the novel and excellent proceedure proposed by J. C. Lozier.

For the sake of simplicity, we have assumed so far that the inherent properties of the servo afford all of the filtering that is necessary to suppress the unwanted modulation terms introduced by the limiter. It is at least conceivable that in practice it may sometimes be necessary to supply additional filtering, by means of supplementary filters. Naturally, whatever effective filters there may be in the system must be such that they pass the wanted signals. This, in combination with our other considerations, implies that the frequency ω_0 must be above the important parts of the Fourier spectra of the signals.

No matter what filtering we may introduce into the system, the output will always contain, as one component, an oscillation having the frequency ω_0 . It is worth remarking that it may not even be desirable to reduce the amplitude of this component below a certain level, by filtering. In fact, such an oscillation furnishes "dynamical lubrication", which diminishes the effects of static friction, backlash, and other parasitic non-linearities tending to degrade the performance of servomechanisms.

We have not said anything very specific about the way in which the oscillation $E_0 \sin \omega_0 t$ is supplied to the limiter; we have merely remarked incidently that it may be supplied by a subsidiary oscillator. Systems in which the oscillation is supplied in that way have certain advantages in the way of flexibility. However, they have the disadvantage of involving a certain amount of extra equipment. We shall close this chapter with a brief description of a variety of oscillating control servomechanisms in which the servomechanism itself is made to supply the oscillation.

Consider the following system:



Suppose that the system is so designed that in the absence of the input signal it oscillates at a frequency ω_0 determined by the phase shifts of the linear elements in the feedback loop. As we have seen, the limiter behaves, as regards the oscillation, as an effectively linear element, having a transmission ratio which decreases as the amplitude is increased. The amplitude of the oscillation adjusts itself so that the amplification around the loop, determined by the amplifications through the limiter and through the linear elements, is unity.

Now suppose that the system is subjected to an input signal. If the corresponding error signal at the input of the limiter is sufficiently small, the amplification of the limiter for the persistent oscillation is substantially unaffected, and the system continues to oscillate at substantially the original frequency and amplitude. As we have seen, the limiter behaves, as regards the signals, as an effectively linear element, having an amplification which is 6 lb. less than the amplification for the persistent oscillation. It is clear that under these conditions we have an oscillating control mechanism, such as we have discussed above. The only novelty in the present situation is the fact that the frequency and amplitude of the persistent oscillation $E_0 \sin \omega_0 t$ are determined by the system itself, instead of being determined independently, as we have tacitly assumed heretofore.

In all of our considerations of the system as a servomechanism, we need only ascribe the proper effective transmission ratio to the limiter, and then proceed in the ways described in the preceding chapters. We do not need to take account explicitly of the persistent oscillation. However, the requirements that the system shall also function as an oscillator imposes certain restrictions on what we can do toward improving its performance as a servomechanism. This can be seen as follows:

Let $F_0(p)$ denote the transmission ratio of the feedback loop for the signals. Then the transmission ratio for the persistent oscillation is $2F_0(p)$; and, by the very fact that the system does oscillate, there is a purely imaginary value of $p = i\omega_0$ for which $2F_0(i\omega_0) = -1$. Hence, the curve, in the Nyquist diagram, described by the running point $w = F_0(p)$ is constrained to pass through the point $w = -\frac{1}{2}$. On the other hand, in order that the performance of the system as a servomechanism shall be satisfactory, the curve must meet the conditions we have discussed in the preceding chapters, including the condition of avoiding the neighborhood of the point $w = -1$. Obviously, the constraint to which the curve is subjected makes it more difficult to meet these conditions than it is in the other systems, where no such constraint exists. In this sense, these self-oscillating servomechanisms are less flexible than are oscillating control servomechanisms in which the oscillation is supplied by an independent generator.

An elementary precaution to be observed, in order that the curve, which is constrained to pass through the point $w = -\frac{1}{2}$, shall avoid the neighborhood of the point $w = -1$, is that the curve should interest the real axis, at the point $w = -\frac{1}{2}$, perpendicularly. This implies that $F_0(p)$ should be varying slowly in amplitude, and rapidly in angle, at the frequency at which the system oscillates.

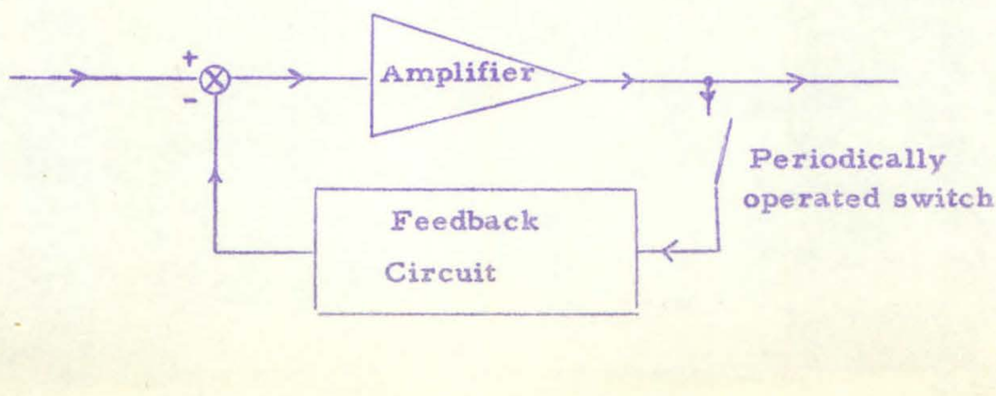
11. Sampling Servomechanisms

All of the servomechanisms that we have considered so far are designed to deal with signals which are given as functions of the continuous variable t. There are situations, however, in which the signals which a servomechanism has to deal with are given as functions of a discrete variable. Such a situation arises, for example, when we have an input signal which has been obtained by determining the values of a function $y_i(t)$ at equally spaced instants 0, t_0, $2t_0$, ... In such a case the input signal is not defined at all in the open intervals between the successive sampling instants.

Naturally, when we have a situation of the kind just described, we are interested in only the values of the output signal at the sampling instants. Consequently, the servomechanism should function so that the corrective effect which it applies to the output signal is governed only by those values, and not by the vaues which the output signal may have during the intervening intervals. A servomechanism which is designed to function in this way may be called a sampling servomechanism. In this chapter we give a brief account of a theory of linear sampling servomechanisms which is very similar, in its point of view and procedure, to the theory of continuously operating servomechanisms that we have been discussing in the proceeding chapters.

Output of a Sampling Circuit

The prototype sampling servomechanism which we shall consider is shown as follows: The system contains the usual forward and feedback circuits. The essential novelty of the system lies in the fact that the



feedback path contains a switch, which is operated periodically so that the feedback loop is closed only during short time intervals located at the equally spaced instants, $0, t_0, 2t_0, \ldots$. The location of the energy-storing, or frequency-selective, elements in the system affects the theory in matters of detail only. Hence, we take the opportunity to simplify the exposition somewhat by assuming that the transmission ratio of the forward circuit is independent of frequency. Then it is essential that the switch be placed in the position shown.

The following analysis is based upon the assumption that the intervals during which the switch is closed are so short that the feedback circuit can be considered to be subjected to a sequence of impulses. It is also based upon the assumption that the response of the feedback circuit to an impulse is a continuous function of time. This last implies that the transmission ratio $F_2(p)$ of the feedback circuit approaches zero at least as rapidly as p^{-2}, as p approaches infinity.

As before, we denote the input and output signals by $y_i(t)$ and $y_o(t)$ respectively. We denote the response of the feedback circuit to an impulse of unit intensity occuring at $t = 0$ by the symbol $H_2'(t)$. Since the Laplace transform of a unit impulse at $t = 0$ is 1. The Laplace transform of the response of the feedback circuit to that impulse is $F_2(p)$. Then according to the inversion formula*

$$H_2'(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_2(p)\, e^{tp}\, dp \tag{11.1}$$

where c is the real constant which is greater than the real part of any pole of $F_2(p)$.

Now it is clear that if the input signal vanishes identically for negative values of t, the value of the output signal at the typical sampling instant nt_0 is given by the formula

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* See for instance Churchill: "Modern Operational Mathematics in Engineering", McGraw Hill (1944), p. 157.

$$y_o(nt_o) = F_1\left[y_i(nt_o) - \theta t_o \sum_{k=0}^{n} y_o(kt_o) H_2'(nt_o - kt_o)\right] \tag{11.2}$$

where F_1 denotes the transmission ratio of the forward circuit, a constant. θ is the fraction of the switching cycle during which the switch is closed.

When $y_i(t)$ and $H_2'(t)$ are known at the sampling instants, the values of $y_o(0)$, $y_o(t_o)$, $y_o(2t_o)$ $\cdots$ can be calculated successively by Eq. (11.2) in an elementary way. However, instead of proceeding in that way, we shall follow a more illuminating course, which will bring the theory of the sampling servomechanism into relation with the ordinary steady-state theory. This approach is due to G. R. Stibitz and C. E. Shannon.

Stibitz-Shannon Theory

Let us write

$$\Phi = \sum_{n=0}^{\infty} y_i(nt_o)\, e^{-nt_o p} \tag{11.3}$$

$$\Psi = \sum_{n=0}^{\infty} y_o(nt_o)\, e^{-nt_o p} \tag{11.4}$$

and

$$F_2^*(p) = t_o \sum_{n=0}^{\infty} H_2'(nt_o)\, e^{-nt_o p} \tag{11.5}$$

These functions are thus periodic functions of p with the imaginary period $2\pi i/t_o$. The functions of nt_o are thus the Fourier coefficient.

For the time being we shall confine our attention to the case in which all of the poles of $F_2(p)$ lie to the left of the imaginary axis. Then the function $H_2'(t)$ ultimately decays exponentially, as t tends toward infinity, and the series in Eq. (11.5) converges for all values of p with real parts which are greater than a certain negative constant. Of course, the convergence of the series in the Eqs. (11.3) and (11.4) depends upon the nature of the input signal. We restrict our attention to input signals for which the series converge in the same manner as the series in Eq. (11.5). This amount only to the mild sort of restriction on $y_i(t)$ that we are accustomed to assume in transient theory.

Multiplying Eq. (11.2) through by $e^{-nt_0 p}$, and then summing over all values of n, we obtain

$$\Psi(p) = F_1\left[\Phi(p) - \theta t_0 \sum_{n=0}^{\infty}{}' e^{-nt_0 p} \sum_{k=0}^{n}{}' y_0(kt_0) H_2'(nt_0 - kt_0)\right]$$

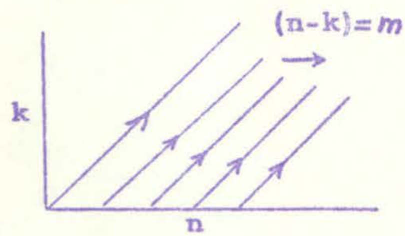
But

$$\theta t_0 \sum_{n=0}^{\infty}{}' e^{-nt_0 p} \sum_{k=0}^{n}{}' y_0(kt_0) H_2'(nt_0 - kt_0)$$

$$= \theta t_0 \sum_{n=0}^{\infty}{}' \sum_{k=0}^{n}{}' y_0(kt_0)\, e^{-kt_0 p}\, e^{-(n-k)t_0 p} H_2'(nt_0 - kt_0)$$

$$= \theta t_0 \sum_{m=0}^{\infty}{}' \sum_{k=0}^{\infty}{}' y_0(kt_0)\, e^{-kt_0 p}\, e^{-mt_0 p} H_2'(mt_0)$$

$$= \theta F_2^*(p) \Psi(p)$$



Therefore

$$\Psi(p) = F_1\left[\Phi(p) - \theta F_2^*(p)\Psi(p)\right]$$

Or

$$\Psi(p) = \frac{F_1 \Phi(p)}{1 + \theta F_1 F_2^*(p)} \tag{11.6}$$

Eq. (11.6) is the analogue of the basic equation for feedback servomechanisms discussed before. What difference there is between the cases lies in the analytical natures of the functions involved. We shall have something to say about this point later.

Now let us assume that the character of $y_0(nt_0)$ is such that the series of $\Psi(p)$ is convergent for values of p with non-negative real part. Then the series is convergent for pure imaginary p. Let $p = is$, then

$$\Psi(is)e^{int_0 s} = \sum_{m=0}^{\infty} y_0(mt_0)e^{-it_0 s(m-n)}$$

Therefore

$$\int_{s_0}^{s_0+\frac{2\pi}{t_0}} \Psi(is)\, e^{int_0 s}\, ds = y_0(nt_0)\frac{2\pi}{t_0}$$

Or

$$y_0(nt_0) = \frac{t_0}{2\pi i}\int_{s_0}^{s_0+\frac{2\pi}{t_0}} \Psi(is)\, e^{int_0 s}\, i\, ds$$

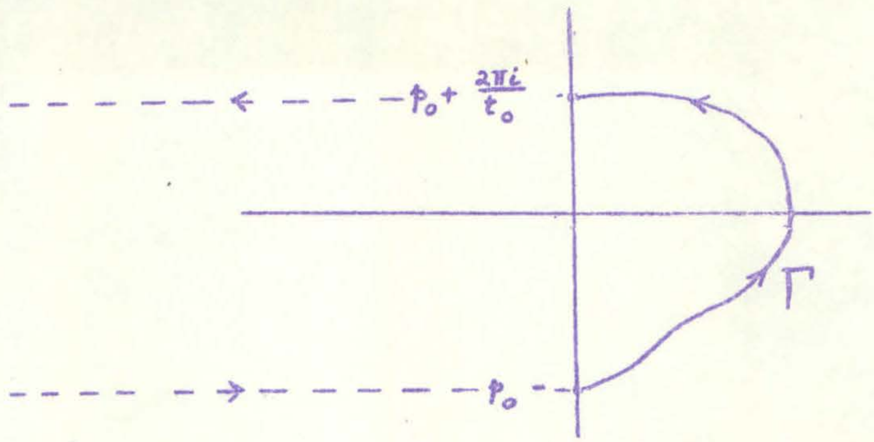
By putting $is = p$, $is_0 = p_0$, we have finally

$$y_0(nt_0) = \frac{t_0}{2\pi i}\int_{p_0}^{p_0+\frac{2\pi i}{t_0}} \Psi(p)\, e^{nt_0 p}\, dp$$

By the Cauchy theorem of complex integration,

$$y_0(nt_0) = \frac{t_0}{2\pi i}\int_{\Gamma} \Psi(p)\, e^{nt_0 p}\, dp = \frac{t_0}{2\pi i}\int_{\Gamma} \frac{F_1 \Phi(p)\, e^{nt_0 p}}{1+\theta F_1 F_2^*(p)}\, dp \tag{11.7}$$

where Γ is a path of integration joining two points separated by the distance $2\pi/t_o$ on the imaginary axis in the p-plane, and passing to the right of all singular points of the integrand. This statement on Γ is now in such a general form that if $\Psi(p)$ has poles with positive real part, Eq. (11.7) is still true.



Now because of the periodicity of Φ (p) and $F_2^*(p)$, we can add to Γ the dotted contours parallel to the real axis. The combined contour then encloses all the poles of the integrand. But it can be shown that for reasonal input, Φ (p) has no poles with positive real parts. Then, in a way very similar to the conventional servomechanism, the necessary and sufficient condition for stability is that the equation

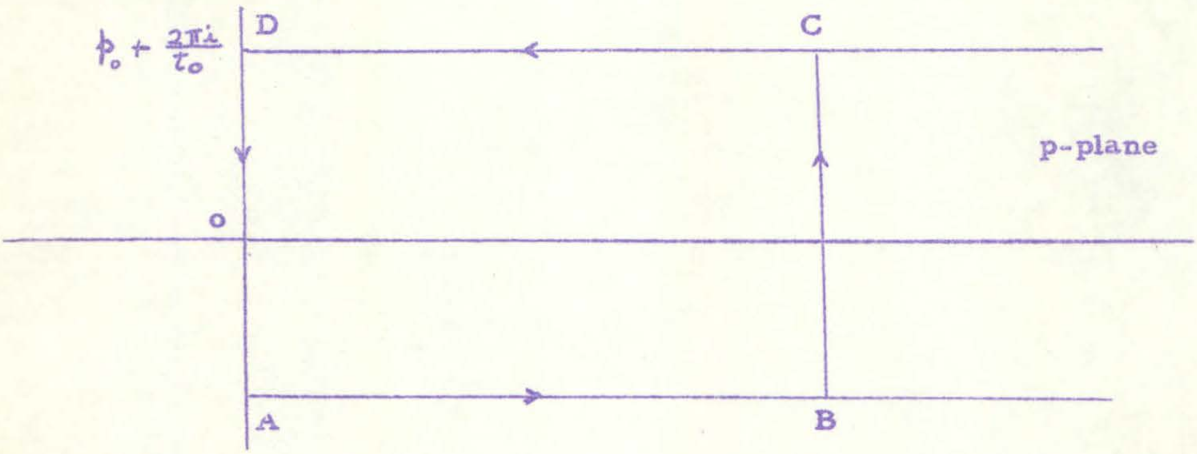
$$1 + \theta F_1 F_2^*(p) = 0 \tag{11.8}$$

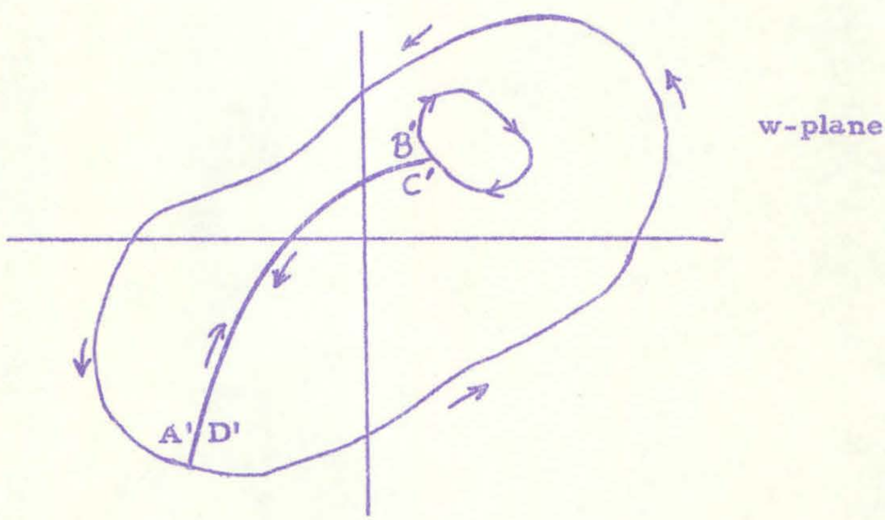
Shall have no roots in the right-hand half of the p-plane. We shall now show how we can implement this condition by an appropriate adoption of Nyquist criterion.

Nyquist Criterion for Sampling Servomechanism

Because of the periodicity of $F_2^*(p)$, it suffices to determine whether or not Eq. (11.8) has any roots in a horizontal half-strip, of width $2\pi/t_o$, extending to the right from the imaginary axis. We are assuming that $F_2^*(p)$ has no singular points on or to the right of the imaginary axis, ~~and we also assume that $F_2^*(p)$ has no singular points on or to the right of the imaginary axis~~, and we also assume that $1 + \theta F_1 F_2^*(p)$ has no zero on the imaginary axis. We can, and do, assume that the half-strip is adjusted vertically so that $1 + \theta F_1 F_2^*(p)$ has no zeros on the horizontal sides.

Now let the point p describe the closed curve ABCDA. Then





the corresponding point

$$w = \theta F_1 F_2^*(p) \tag{11.9}$$

describes a certain closed curve A'B'C'D'A' in the w-plane. We do not try to show the curve described by w realistically. When p describes AB, w describes the arc A'B'. When p describes BC, w describes an arc B'C'; and, because of the periodicity of $F_2^*(p)$, B'C' is a closed cure. When p describes CD, w describes an arc C'D'; and, because of the periodicity of $F_2^*(p)$, C'D' coincides, except for sense, with A'B'. Finally, when p describes DA, w describes an arc D'A', which is a closed curve.

By Cauchy's theorem, Eq. (11.8) does, or does not, have roots in the rectangle ABCD according as the radius rector from the point w = - 1 to the running point Eq. (11.9) does, or does not, make a non-zero net number of revolutions as the running point describes the curve A'B'C'D'A'.

Now consider what happens when the side BC of the rectangle recedes to infinity. It is easy to see from Eq. (11.5) that the closed curve formed by the arc B'C' shrinks to a single point. Therefore the effective contour of w is the arc D'A'. Obviously, Eq. (11.8) does, or does not, have roots in the half-strip according as the radius rector from the point w = - 1 to the contour does, or does not, make a non-zero net number of revolutions as the limiting curve is described.

We have now given the fundamentals of the Stibitz-Shannon theory of sampling servomechanisms which is very similar, both in its point of view and in its form, to the theory of servomechanism with continuous operation.

Steady State Error

If the input is a step function of magnitude a,

$$\Phi(p) = a\sum_{n=0}^{\infty} e^{-nt_0 p} = a\sum_{n=0}^{\infty}\left(e^{-t_0 p}\right)^n = \frac{a}{1-e^{-t_0 p}}$$

Then according to Eq. (11.7)

$$y_o(nt_o) = \frac{at_oF_1}{2\pi i}\int_{\Gamma}\frac{e^{nt_op}\,dp}{[1-e^{-t_op}][1+\beta F_1F_2^*(p)]}$$

As $n \to \infty$, the only pole of importaance is at the origin,

$$\lim_{n\to\infty} y_o(nt_o) = \frac{aF_1}{1+\beta F_1F_2^*(o)}$$

Therefore the condition for small steady state error is

$$F_1 \cong 1+\beta F_1F_2^*(o)$$

Or

$$F_1 \cong \frac{1}{1-\beta F_2^*(o)} \qquad (11.10)$$

This gives the approximate magnitude of the gain for the forward circuit.

<u>Calculation of $F_2^*(p)$</u>

We have one more step to take before our theory of sampling servomechanisms can be regarded as being of much practical value. Both of the functions $F_2(p)$ and $F_2^*(p)$ serve to characterize the feedback circuit. $F_2(p)$ is the significant characteristic when the circuit is used as part of a continuously operating servomechanism; and $F_2^*(p)$ is the significant characteristic when the circuit is used as part of a sampling servomechanism. The familiar function $F_2(p)$ is mathematically much the simpler of the two, and furthermore it is the function which is used directly in our common techniques for designing circuits. It is important, therefore, that we relate $F_2^*(p)$ to $F_2(p)$, and in as direct a manner as possible.

Assuming that the real part of p is greater than c which according to our assumptions is a negative number, we have by Eqs. (11.1) and (11.5),

$$F_2^*(p) = \frac{t_0}{2\pi i}\sum_{n=0}^{\infty} e^{-nt_0p}\int_{c-i\infty}^{c+i\infty} F_2(q)\,e^{nt_0q}\,dq$$

$$= \frac{t_0}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_2(q)\,dq\sum_{n=0}^{\infty} e^{-nt_0(p-q)}$$

$$= \frac{t_0}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{F_2(q)\,dq}{1-e^{-t_0(p-q)}} \tag{11.11}$$

We proceed to evaluate the right-hand member of (11.11) by the method of residues.

The integrand has certain poles, the poles of $F_2(p)$, lying to the left of the path of integration, and certain poles, the roots of the equation $1-e^{-t_0(p-q)}=0$, lying to the right of the path of integration. It is easily seen that the integration upward along the line

Real part of $q = c$

is equivalent to integration in the negative sense along the closed contour formed by that line and the infinite semicircle in the right-hand half-plane. Hence the right-hand member of (11.11) is $-t_0$ times the sum of the residues of the integrand with respect to the several roots of the equation $1-e^{-t_0(p-q)}=0$.

Now the typical root of the equation is $q = p + 2\pi in/t_0$, where n is an integer, and the residue of the intagrand with respect to that pole is $-F_2(p + 2\pi in/t_o)/t_o$. Therefore finally

$$F_2^*(p) = \sum_{n=-\infty}^{n=\infty} F_2(p + 2\pi in/t_o), \quad \mathcal{R}\ell\, p > c \tag{11.12}$$

This formula gives considerable insight into the properties of $F_2^*(p)$, and at times may be useful in making approximate calculations. However, we can easily obtain an exact representation of $F_2^*(p)$ in finite form.

The function $F_2(p)$ can be represented as the sum of a finite number of partial fractions, thus:

$$F_2(p) = \sum_{k=1}^{K} \frac{a_k}{p - p_k} \tag{11.13}$$

where the a_k's and the p_k's are constants. Consequently, we can write by using Eq. (11.12),

$$F_2^*(p) = \sum_{k=1}^{K} a_k \left\{ \frac{1}{p - p_k} + \sum_{n=1}^{\infty} \left[\frac{1}{\frac{2\pi in}{t_o} + (p - p_k)} - \frac{1}{\frac{2\pi in}{t_o} - (p - p_k)} \right] \right\}$$

$$= \sum_{k=1}^{K} a_k \left[\frac{1}{p - p_k} + \sum_{n=1}^{\infty} \frac{2(p - p_k)}{4\pi^2 n^2/t_o^2 + (p - p_k)^2} \right] \tag{11.14}$$

Now it is known that

$$\sum_{n=1}^{\infty} \frac{2(p - p_k)}{4\pi^2 n^2/t_o^2 + (p - p_k)^2} = \frac{(p - p_k)t_o^2}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2 + (p - p_k)^2 t_o^2/4\pi^2}$$

$$= \frac{(p - p_k)t_o^2}{2\pi^2} \left[-\frac{2\pi^2}{(p - p_k)^2 t_o^2} + \frac{\pi^2}{(p - p_k)t_o} \coth \frac{(p - p_k)t_o}{2} \right]$$

Therefore Eq. (11.14) becomes

$$F_2^*(p) = \frac{t_0}{2} \sum_{k=1}^{K} a_k \coth \frac{(p-p_k)t_0}{2} \tag{11.15}$$

By means of this formula we can compute $F_2^*(p)$ exactly for any value of p.

When t_0 is so small that $F_2(i\omega)$ is neglibibly small outside of the interval $-\pi/t_0 < \omega < \pi/t_0$, the qualitative nature of $F_2^*(i\omega)$ is immediately apparent from Eq. (11.12). In fact, $F_2^*(i\omega)$ is approximately equal, in the interval $-\pi/t_0 < \omega < \pi/t_0$, to the function $F_2(i\omega)$. We shall now see that when t_0 is large we can obtain an equally simple approximation to $F_2^*(i\omega)$.

Referring to Eq. (11.15), let us write

$$p_k = -\alpha_k + i\omega_k \tag{11.16}$$

where α's and ω's are all real. In accordance with our assumption, the α's are all positive.

Now we have

$$F_2^*(i\omega) = \frac{t_0}{2} \sum_{k=1}^{K} a_k \coth \frac{t_0}{2}[\alpha_k + i(\omega - \omega_k)]$$

$$= \frac{t_0}{2} \sum_{k=1}^{K} a_k \frac{1 + e^{-t_0[\alpha_k + i(\omega-\omega_k)]}}{1 - e^{-t_0[\alpha_k + i(\omega - \omega_k)]}}$$

and hence, when $t_0 >> 1$,

$$F_2^*(i\omega) \cong \frac{t_0}{2} \sum_{k=1}^{K} a_k \left\{ 1 + 2 e^{-t_0[\alpha_k + i(\omega - \omega_k)]} \right\} \tag{11.17}$$

When p is large, Eq. (11.13) can be written as

$$F_2(p) = \frac{1}{p}\sum_{k=1}^{K}{}' a_k + \frac{1}{p^2}\sum_{k=1}^{K}{}' a_k p_k + \text{-----}$$

But we have assumed as a condition for continuous response of the feedback circuit to an impulse that $F_2(p) \sim \frac{1}{p^2}$ when p is large. Therefore

$$\sum_{k=1}^{K}{}' a_k = 0 \tag{11.18}$$

Then Eq. (11.17) becomes

$$F_2^*(i\omega) \cong t_0\, e^{-it_0\omega} \sum_{k=1}^{K}{}' a_k\, e^{t_0 p_k} \tag{11.19}$$

For physical systems, p_k's are real or form pairs of complex conjugates. Therefore the finite sum in Eq. (11.19) is actually real. Then the graph of $F_2^*(i\omega)$ as $-\frac{\pi}{t_0} < \omega\ \frac{\pi}{t_0}$ is a circle with the radius

$$t_0 \sum_{k=1}^{K}{}' a_k\, e^{t_0 p_k}$$

<u>Comparison of Continuously Operating and Sampling Servomechanism</u>

For small t_o, we have seen that $F_2^*(i\omega)$ is approximately $F_2(i\omega)$. The stability criterion for the continuously operating servomechanism is that the contour $F_1F_2(i\omega)$ should avoid the point -1. For the sampling servomechanism, the contour $\theta F_1F_2^*(i\omega)$ should avoid the point -1, or the contour $F_1F_2(i\omega)$ should avoid the point $1/\theta$. Therefore if stability is the only consideration, the sampling servomechanism can have much larger gain than the conventional servomechanism.

For large values of t_o, because of Eq. (11.19), the Nyquist criterion becomes simply

$$\theta F_1 \left| t_0 \sum_{k=1}^{K}{}' a_k e^{t_0 p_k} \right| < 1$$

Since p_k has negative real part, the radius of $F_2^*(i\omega)$ contour is very small. This fact together with the smallness of θ, allows very large gain for the forward circuit without instability.

Pole of $F_2(p)$ at Origin

In practice, the function $F_2(p)$ is quite likely to have a pole at $p = o$. So far we have excluded this case from consideration, in order to avoid having to deal with certain minor complications. However, we shall now consider the case briefly.

In the first place, we observe that when $p = o$ is a pole of $F_2(p)$ the constant c must be positive, and that our representation of $F_2^*(p)$ in terms of infinite series are only valid for values of p with positive real parts.

In the second place, the Nyquist diagram for the system also undergoes changes. Specifically, instead of getting an actual closed curve, we get an open curve, the ends of which are to be regarded as being joined by an infinite semi-circle.

Our representation of $F_2^*(p)$ in finite form remain valid. If we set $p_1 = o$, then Eq. (11.15) gives

$$F_2^*(i\omega) = \frac{t_0}{2}\left[-ia_1 \cot\frac{\omega t_0}{2} + \sum_{k=2}^{K}{}' a_k \coth\frac{t_0[\alpha_k + i(\omega-\omega_k)]}{2}\right]$$

For t_o large,

$$F_2^*(i\omega) = \frac{t_0}{2}\left[-ia_1 \cot\frac{\omega t_0}{2} + \sum_{k=2}^{K}{}' a_k \left\{1 + 2e^{-t_0[\alpha_k + i(\omega-\omega_k)]}\right\}\right]$$

But

$$a_2 + a_3 + \cdots + a_k = -a_1$$

So

$$F_2^*(i\omega) = -\frac{a_1 t_0}{2}\left[1 + i\cot\frac{\omega t_0}{2}\right] + t_0 e^{-it_0\omega} \sum_{k=2}^{K}{}' a_k e^{t_0 p_k} \qquad (11.20)$$

The constant a_1 is, of course, real and positive. Hence the Nyquist diagram is a sinuous variation of a straight line.

10. Control and Stability of Systems with Time Lag

In the previous chapters, we have discussed systems with increasing degrees of complexity. But the systems are all linear with constant coefficients. Linear systems of next order of difficulty in their analysis are systems involving time lag. Such systems have been studied by many authors* from various points of view. But our interest is in the possibility of stabilizing the system and thus remove the danger of combusiton instability occuring in jet propulsion power plants. For instance, it is known** that the chucking or rough burning in rocket engine is due to the instability caused by ignition or combustion time lag. We shall study in this chapter the possibility of stabilization by a proper feed-back control.

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* A. Callender, D. Hartree and A. Porter, "Time Lag in a Control System" Phio. Trans. Royal Society of London (A), 235: 415-444 (1936).

N. Minosky, "Self-excited Oscillations in Dynamical Systems Possessing Retarted Actions", J. Appl. Mechanics, 9: 67-71 (1942).

H. I. Ansoff, "Stability of Linear Oscillating Systems with Constant Time Lag", J. Appl. Mechanics, 16: 158-164 (1949).

***D. F. Gunder and D. R. Friant, "Stability of Flow in a Rocket Motor" J. Appl. Mechanics 17: 327-333 (1950).

M. Yachter, Discussion of above paper, Ibid, 18:114-116 (1951).

M. Summerfield, "A Theory of Unstable Combustion in Liquid Propellant Rocket Motors", J. ARS, 21: 108-114 (1951).

L. Crocco, "Aspects of Combustion Stability in Liquid Propellant Rocket Motors", J. ARS, 21: 163-178 (1951).

Time Lag in Combustion

The basic concept in this discussion originated from L. Crocco. He assumes that the rate at which the injected propellant is prepared for the final rapid reaction is a function of the prvailing pressure in the rocket chamber. This is certainly true for processes such as atomization, evaporation, heat transfer, etc. Therefore if t is the time instant at which the propellant becomes hot gas and $t-\tau$ is the time instant this parcel of propellant was injected, then

$$\text{Constant} = \int_{t-\tau}^{t} f(p)\,dt' \tag{10.1}$$

where $f(p)$ is the time rate of "preparation for combustion", and p is the prevailing chamber pressure at the instant t'.

If the burning rate in the interval of time dt is called $\dot{m}_b(t)$, the injection rate $\dot{m}_i(t)$ and if the time lag is $\tau(t)$, the mass burning during the time from t to t + dt must be equal to the mass injected during the time from $t-\tau$ to $t-\tau+d(t-\tau)$. Therefore

$$\dot{m}_b(t)\,dt = \dot{m}_i(t-\tau)\,d(t-\tau) \tag{10.2}$$

The mass of hot gas stored in the chamber is $M_g(t)$ and is proportional to the pressure $p(t)$, assuming constant temperature of combustion. Thus if $(\bar{\ })$ denotes the steady-state non-oscillatory quanities, then

$$\frac{p(t)}{\bar{p}} = \frac{M_g(t)}{\bar{M}_g} \tag{10.3}$$

For low frequency oscillations, the flow in the nozzle can be considered as quasi-stationary, and the rate of mass discharge $\dot{m}_e$ can be taken as proportional to the pressure p. Thus

$$\frac{\dot{m}_e(t)}{\bar{\dot{m}}} = \frac{p(t)}{\bar{p}} \tag{10.4}$$

where $\bar{\dot{m}}$ is the steady mass flow rate through the rocket chamber.

The mass balance of gas then gives

$$\dot{m}_b\, dt = \dot{m}_e\, dt + dM_g \tag{10.4a}$$

This equilibrium assumes, of course, that the unburned propellant does not occupy any volume and thus does not enter into the continuity equation. Equations (10.1) to (10.4a) specify completely the combustion process in the chamber. By combining (10.4a) with (10.2), we have

$$\frac{dM_g}{dt} + \dot{m}_e = \dot{m}_b = \dot{m}_i(t-\tau)\left(1-\frac{d\tau}{dt}\right)$$

Now by introducing the non-dimensional variables φ, μ and μ_b as the fractional non-steady quantities, i.e.,

$$\varphi = \frac{p-\bar{p}}{\bar{p}}, \quad \mu = \frac{\dot{m}_i - \bar{\dot{m}}}{\bar{\dot{m}}}, \quad \mu_b = \frac{\dot{m}_b - \bar{\dot{m}}}{\bar{\dot{m}}} \tag{10.5}$$

we have, because of (10.3) and (10.4),

$$\frac{\bar{M}_g}{\bar{\dot{m}}}\frac{d\varphi}{dt} + \varphi + 1 = \mu_b + 1 = \left(1-\frac{d\tau}{dt}\right)\left[\mu(t-\tau)+1\right] \tag{10.6}$$

It remains to calculate $d\tau/dt$ from (10.1). By differentiating (10.1) with respect to t,

$$0 = f[p(t)] - f[p(t-\tau)]\left(1-\frac{d\tau}{dt}\right)$$

Now the pressure variations will be assumed to be small to linearize our system, i.e.,

$$f[p(t)] = f(\bar{p}) + \left(\frac{df}{dp}\right)_{p=\bar{p}} \bar{p}\,\varphi(t)$$

$$f[p(t-\tau)] = f(\bar{p}) + \left(\frac{df}{dp}\right)_{p=\bar{p}} \bar{p}\,\varphi(t-\tau)$$

Thus

$$1-\frac{d\tau}{dt}=\frac{f[\bar{p}(t)]}{f[p(t-\tau)]}\cong 1+\left\{\bar{p}\frac{d}{d\bar{p}}[\mathrm{Log} f]\right\}[\varphi(t)-\varphi(t-\tau)]$$

Let us put

$$\bar{p}\frac{d}{d\bar{p}}[\mathrm{Log} f]=\frac{d\,\mathrm{Log} f}{d\,\mathrm{Log}\,\bar{p}}=n \tag{10.7}$$

With these relations, the linearized form of (10.6) is

$$\frac{d\varphi}{dz}+\varphi=\mu_b=\mu(z-\delta)+n[\varphi(z)-\varphi(z-\delta)] \tag{10.8}$$

where $\quad z=t/\theta_g, \quad \delta=\tau/\theta_g, \quad \theta_g=\bar{M}_g/\bar{\dot{m}}$ (10.9)

The τ and δ are now constants as a result of linearization and is equal to the values at the steady-state conditions. θ_g is easily shown to be gas residence time given by

$$\theta_g=\frac{L^*c^*}{RT_g} \tag{10.10}$$

where L^* and c^* are the characteristic length and the characteristic velocity respectively. (10.8) is the fundamental equation of the combustion chamber.

Intrinsic Instability

Let us study first the condition for intrinsic instability, i.e., instability without the coupling effects of the feed system. In this case, the injection is assumed to be independent of chamber conditions, i.e., $\mu\equiv 0$. Then (10.8) reduses to

$$\frac{d\varphi}{dz}+(1-n)\varphi+n\varphi(z-\delta)=0 \tag{10.11}$$

Now let $\varphi = e^{pz}$ (10.12)

then

$$[p+(1-n)+ne^{-p\delta}]e^{pz} = 0$$

The problem then revolves to find whether

$$p+(1-n)+ne^{-p\delta} = 0 \qquad (10.13)$$

has roots whose real part is positive. We can write (10.13) as

$$G(p) = e^{-\delta p} - \left\{-\frac{1-n}{n} - \frac{1}{n}p\right\} = g_1(p) - g_2(p)$$

where

$$g_1(p) = e^{-\delta p}$$

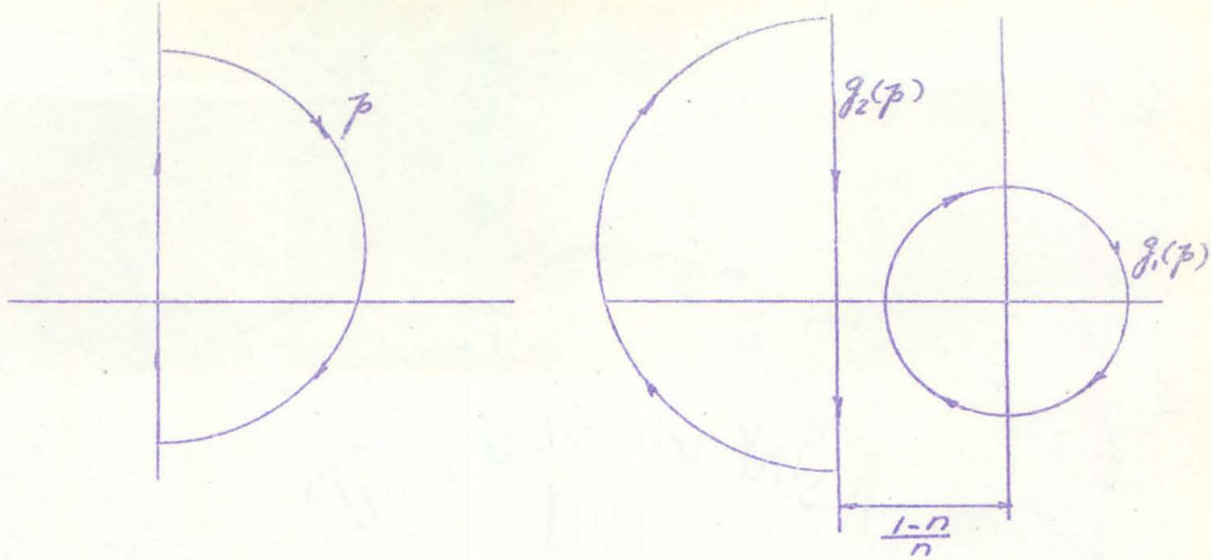
and

$$g_2(p) = -\frac{1-n}{n} - \frac{1}{n}p$$

Now the question whether $G(p) = 0$ has a root of positive real part is answered by making p to take a contour enclosing the positive half of p-plane, and then see whether the vector $G(p)$ makes any complete revolutions. Now the vector $G(p)$ is the vector with vertex on $g_1(p)$ and the starting point on $g_2(p)$. When p makes the contour indicated, $g_1(p)$ is a curve within the unit circle. $g_2(p)$ is a straight line and half-circle in the negative half of p-plane. It is evident that if

$$0 < n < \frac{1}{2}$$

the $g_1(p)$ curve and $g_2(p)$ lies completely outside of each other and therefore $G(p)$ cannot make complete revolutions. The system is thus stable.



If $n > \frac{1}{2}$, then the curves can partially cover each other. But if the point P on $g_1(p)$ is to the right of the straight portion of $g_2(p)$ enclosed by the unit circle, the vector $G(p)$ cannot make a complete revolution. The system is again stable. Thus if $n > \frac{1}{2}$ and if

$$\cos \delta \sqrt{2n-1} > -\frac{1-n}{n}$$

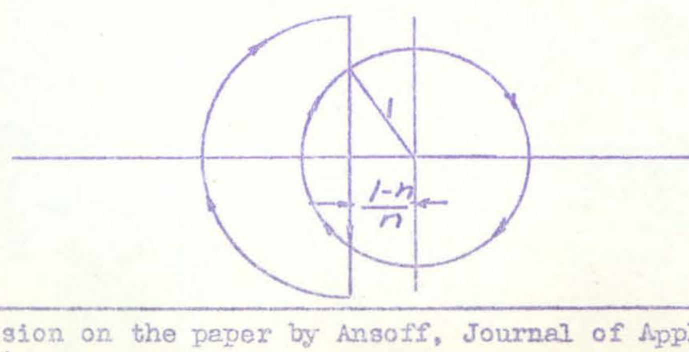
the system is stable. This means for stability with $n > \frac{1}{2}$,

$$\delta < \delta^* \tag{10.14}$$

where

$$\delta^* = \frac{\cos^{-1}\left(-\frac{1-n}{n}\right)}{\sqrt{2n-1}} = \frac{1}{\sqrt{2n-1}}\left(\pi - \cos^{-1}\frac{1-n}{n}\right) \tag{10.15}$$

These results were obtained by L. Crocco. Our derivation, however, is different from his, by following the method of M. Satche.* The diagrams of $g_1(p)$ and $g_2(p)$ may be called Satche diagram.



* M. Satche, Discussion on the paper by Ansoff, Journal of Appl. Mech. 16: 419-420 (1949).

When $\delta = \delta^*$, then

$$G(i\omega^*) = g_1(i\omega^*) - g_2(i\omega^*) = 0$$

where

$$\omega^* = \sqrt{2n-1} \tag{10.16}$$

Thus when $\delta = \delta^*$, a solution of (10.11) is sinusoidal oscillation of angular frequency ω^*/θ_g with ω^* determined by (10.16). For n near 1/2, the frequency will be very low.

Propellant Feed System

Let us consider the propellant to be fed by a centrifugal pump run at constant speed. Then if the time characteristics of the transient is not too short, we can assume that quasi-steady condition exist and the relation between mass flow rate $\dot{m}_0$ and the pressure p_0 is given by

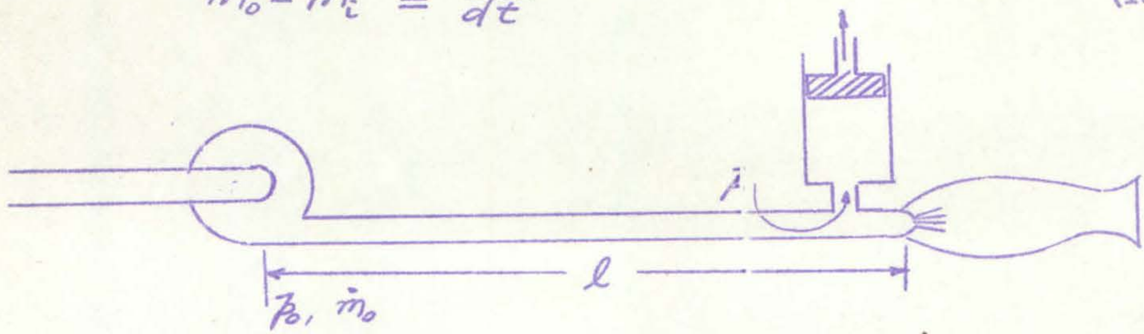
$$\frac{\dot{m}_0 - \bar{\dot{m}}}{\bar{\dot{m}}} = -\alpha\left(\frac{p_0 - \bar{p}_0}{\bar{p}_0}\right) \tag{10.17}$$

Let us for simplicity neglect the frictional and other loss in the pipe. Then under steady state

$$\bar{p}_0 - \bar{p} = \overline{\Delta p} = \frac{\bar{\dot{m}}^2}{2\rho A_i^2} \tag{10.18}$$

where A_i represents the area of the jets from the injector, i.e., the orifice area corrected for the contraction coefficient. We shall neglect the elasticity of the propellant line, but we shall consider a controlled reservoir off the line just ahead of the injector. The capacity C of the reservoir varies with time. Then

$$\dot m_o - \dot m_i = \frac{dc}{dt} \tag{10.19}$$



Now let p_1 be the pressure at the reservoir. Then the work done by the instantaneous pressure drop $p_o - p_1$ must be equal to the increase of kinetic energy in the connecting line of length l, and area A_c. Thus

$$\frac{\dot m_o}{\rho}(p_o - p_1) = \frac{1}{2}\rho l A_c \frac{d(V_c^2)}{dt} \tag{10.20}$$

where V_c is the flow velocity in the connecting line. So.

$$\rho A_c V_c = \dot m_o \tag{10.21}$$

If we neglect the small propellant mass between the reservoir and injector, then

$$p_1 - p = \frac{1}{2}\frac{\dot m_i^2}{\rho A_i^2} \tag{10.22}$$

By combining (10.20), (10.21) and (10.22), we have

$$p_o - p = \frac{1}{2}\frac{\dot m_i^2}{\rho A_i^2} + \frac{l}{A_c}\frac{d\dot m_o}{dt}$$

When we subtract (10.18) from the above equation, we have

$$(p - \bar p_o) - (p - \bar p) \cong 2(\Delta \bar p)\mu + \frac{l}{A_c}\frac{d}{dt}\left[\dot m_i + \frac{dc}{dt}\right]$$

But by (10.17) and (10.19),

$$-\frac{1}{\alpha}\left(1+\frac{\Delta\bar{p}}{\bar{p}}\right)\left[\mu+\frac{1}{\bar{\dot{m}}\theta_g}\frac{dc}{dz}\right]-\varphi = 2\left(\frac{\Delta\bar{p}}{\bar{p}}\right)\mu+\frac{l}{A_c}\frac{\bar{\dot{m}}}{\bar{p}\theta_g}\frac{d}{dz}\left[\mu+\frac{1}{\bar{\dot{m}}\theta_g}\frac{dc}{dz}\right]$$

Or introduce a non-dimensional reservoir capacity K , and constants J and P

$$\left.\begin{aligned} K &= c/\bar{\dot{m}}\theta_g \\ J &= \frac{l\,\bar{\dot{m}}}{2\Delta\bar{p}A_c\theta_g} \\ P &= \frac{\bar{p}}{2\Delta\bar{p}} \end{aligned}\right\} \tag{10.23}$$

Then

$$-\frac{1}{\alpha}\left(P+\frac{1}{2}\right)\left[\mu+\frac{dK}{dz}\right]-P\varphi=\mu+J\frac{d}{dz}\left[\mu+\frac{dK}{dz}\right] \tag{10.24}$$

This is the feed-system equation when the external reservoir control K is specified. Together with (10.8), it determines the transient behavior of the system.

<u>Feed-back Control</u>

We shall close the control loop, if we make the reservoir capacity to depend upon φ . That is we measure the instantaneous chamber pressure p by a pressure pickup and this measurement is made to act on K through a proper amplifier and servo. Thus

$$F\left(\frac{d}{dz}\right)\varphi = K \tag{10.25}$$

where $F(\frac{d}{dz})$ is a ratio of polynomials in $\frac{d}{dz}$, with the denominator having a order higher than the numerator. Then let us substitute

$$\varphi = Ae^{pz}$$
$$\mu = Be^{pz}$$
$$K = De^{pz}$$

We have from (10.8)

$$[p+(1-n)+ne^{-\delta p}]A + [-e^{-\delta p}]B + [0]D = 0$$

$$[P]A + [1+\tfrac{1}{\alpha}(P+\tfrac{1}{2})+Jp]B + [\tfrac{1}{\alpha}(P+\tfrac{1}{2})p + Jp^2]D = 0$$

$$F(p)A + \qquad [0]B - \qquad D = 0$$

The appropriate p is thus determined by the equation

$$\begin{vmatrix} p+(1-n)+ne^{-\delta p} & -e^{\delta p} & 0 \\ P & 1+\frac{1}{\alpha}(P+\frac{1}{2})+Jp & \frac{1}{\alpha}(P+\frac{1}{2})p+Jp^2 \\ F(p) & 0 & -1 \end{vmatrix} = 0$$

Or

$$\{p+(1-n)\}[\{1+\tfrac{1}{\alpha}(P+\tfrac{1}{2})\}+Jp] + e^{-\delta p}[\{n+\tfrac{n}{\alpha}(P+\tfrac{1}{2})+P\}+p\{nJ+F(p)(\tfrac{P+\frac{1}{2}}{\alpha}+Jp)\}] = 0 \quad (10.26)$$

For the system to be stable, the roots p of (10.26) should never have positive real parts. The problem of stabilization by feed-back control is then the question of designing the transfer function $F(p)$ such that the condition of stable system is indeed satisfied.

Problem 10.1

Derive the equation for the exponent p when the feed-system has an additional spring-loaded reservoir midway between the pump and injector.

Stabilization by Feed-back

To study the possibility of stabilizing the system through the feed-back transfer function $F(p)$, let us first analyse the situation without feed-back, i.e., $F(p)=0$. Then (10.26) becomes

$$(p+p_1)(p+p_2)+ne^{-\delta p}(p+p_3)=0 \tag{10.27}$$

where

$$\left.\begin{aligned} p_1 &= (1-n) \\ p_2 &= \frac{1}{J}\left(1+\frac{P+\frac{1}{2}}{\alpha}\right) \\ p_3 &= \frac{1}{J}\left(1+\frac{P+\frac{1}{2}}{\alpha}+\frac{P}{n}\right) \end{aligned}\right\} \tag{10.28}$$

Therefore $$p_3-p_2=\frac{P}{nJ}>0 \tag{10.29}$$

Since all $p's$ are positive, we shall not alter the situation in the right half of the p-plane by dividing (10.27) by $p+p_3$. Then

$$G(p)=e^{-\delta p}-\left(-\frac{1}{n}\right)\frac{(p+p_1)(p+p_2)}{p+p_3}=0$$

Let us put

$$g_1(p)=e^{-\delta p}$$

and

$$g_2(p)=-\frac{1}{n}\frac{(p+p_1)(p+p_2)}{p+p_3}$$

The $g_1(p)$ is the same "unit circle" mentioned in the previous section, but now $g_2(p)$ is not a straight line parallel to the imaginary axis for along the imaginary axis, it is somewhat deformed: Let

$$p=i\omega$$

$$g_2(p)=g_2(i\omega)=\frac{-\omega^2[p_1-(p_3-p_2)]+p_1p_2p_3}{n(\omega^2+p_3^2)}-i\omega\frac{\omega^2+p_1(p_3-p_2)+p_2p_3}{n(\omega^2+p_3^2)}$$

and let us take $0 < n < 1$.

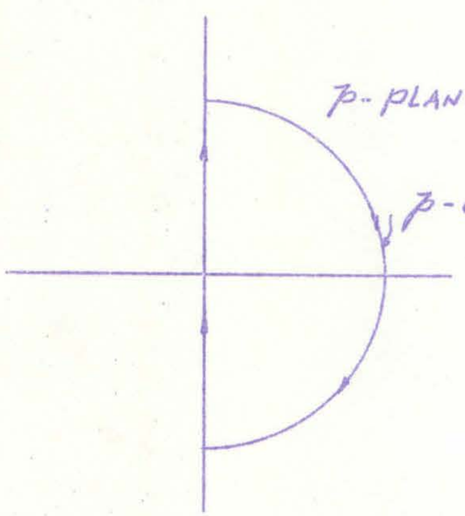
When $\omega = 0$, $g_2(i\omega)$ is real,

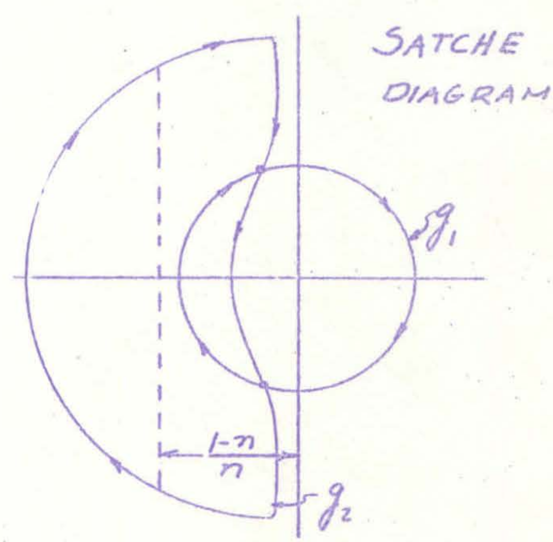
$$g_2(0) = -\frac{1}{n} p_1 \left(\frac{p_2}{p_3}\right) = -\frac{1-n}{n}\left(\frac{p_2}{p_3}\right)$$

Since $p_3 > p_2$ is indicated by (10.29), $g_2(i\omega)$ crosses the real axis at a point to the right of the curve for the "intrinsic" case. When $\omega \to \infty$, the real part of g_2 is

$$\lim_{\omega \to \infty} R g_2(i\omega) = -\frac{1}{n}\left[p_1 - (p_3 - p_2)\right]$$
$$= -\left[\frac{1-n}{n} - \frac{P}{n^2 J}\right]$$

The absolute value of this is again smaller than $1-n/n$. Therefore the whole g_2 contour is displaced towards the right. The effect of the feed-system is then to bring the g_2 contour closer to the unit circle of the g_1 contour. The general effect of the feed system is then de-stabilizing.





When the g_1 contour and the g_2 contour intersect each other, then instability can occur for large enough values of δ. The critical value of δ, δ^*, can be determined from a consideration of the point of intersection of g_1 and g_2.

When $F(p)\neq 0$, (10.26) can be written as

$$e^{-\delta p}\left[1+\frac{pF(p)}{n}\,\frac{p+\frac{P+\frac{1}{2}}{2I}}{p+p_3}\right]-\left(-\frac{1}{n}\right)\frac{(p+p_1)(p+p_2)}{(p+p_3)}=0 \qquad (10.30)$$

where the constants p_1, p_2 and p_3 are given by (10.28) and are all positive. If the quantity within the square bracket has no zeros with positive real part, then we can write again

$$G(p)=g_1(p)-g_2(p)$$

with

$$g_1(p)=e^{-\delta p}$$

and

$$g_2(p)=-\frac{1}{n}\,\frac{(p+p_1)(p+p_2)}{(p+p_3)}\,\frac{1}{1+\frac{pF(p)}{n}\frac{\left(p+\frac{P+\frac{1}{2}}{2I}\right)}{(p+p_3)}}$$

The stability condition is again satisfied if the contour traced by $g_2(p)$ lies completely outside of the unit circle, the contour of $g_1(p)$. This can be accomplished by a proper choice of $F(p)$:

As an example, let

$$F(p)=-K\frac{1}{p(p+p_4)}$$

where p_4 and K are real and positive. It is then evident that the behavior of $g_2(p)$ at large p is not altered by the introduction of $F(p)$. The stability condition can thus be accomplished by only pushing the $g_2(p)$ contour towards the left for small values of $p=i\omega$. Before we discuss this condition, we must first put down the conditions for no zeros of positive real part for the polynomial within the square bracket of (10.30). This polynomial is

$$\frac{p^2+[p_3+p_4-\frac{K}{n}]p+[p_3 p_4-\frac{K}{n}\frac{P+\frac{1}{2}}{\alpha J}]}{(p+p_4)(p+p_3)}$$

Therefore the conditions can be written as

$$p_3+p_4>\frac{K}{n} \tag{10.31}$$

and

$$\begin{aligned} 0 &\geq (p_3+p_4-\frac{K}{n})^2-4[p_3 p_4-\frac{K}{n}\frac{P+\frac{1}{2}}{\alpha J}] \\ &= (p_3-p_4)^2+\frac{K^2}{n^2}-2\frac{K}{n}(p^3+p_4)+4\frac{K}{n}\frac{P+\frac{1}{2}}{\alpha J} \end{aligned} \tag{10.32}$$

Let us take $p_4 = p_3$, then (10.31) and (10.32) can be written as

$$2p_3>\frac{K}{n}$$

and

$$\frac{K}{n}-4\,\frac{1+\frac{P}{n}}{J}\leq 0$$

If we take

$$\frac{K}{n}=2.8\,\frac{1+\frac{P}{n}}{J}$$

then the second condition is satisfied. The first condition then becomes

$$\frac{2P+1}{\alpha}>0.8(1+\frac{P}{n})$$

This condition is always satisfied if $\alpha=1$ and $n=\frac{1}{2}$. For the following discussion, we shall assume this to be true. Then

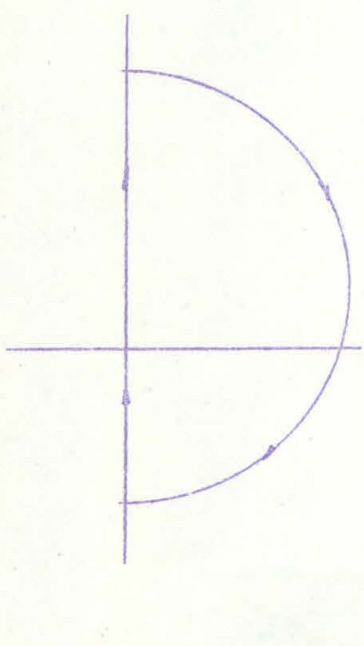
$$g_2(0)=-\frac{P+\frac{3}{2}}{3P+\frac{3}{2}}\cdot\frac{1}{1-2.8\,\frac{(2P+1)(P+\frac{1}{2})}{(3P+\frac{3}{2})^2}}$$

The first factor is the value of $g_2(0)$ when the feed-back is absent, it is always less than 1 in magnitude. The system is thus conditionally unstable, i.e., it will become unstable with large enough lag δ. But with the chosen feed-back, $g_2(0)$ is larger than 1 in magnitude if $P<7$.

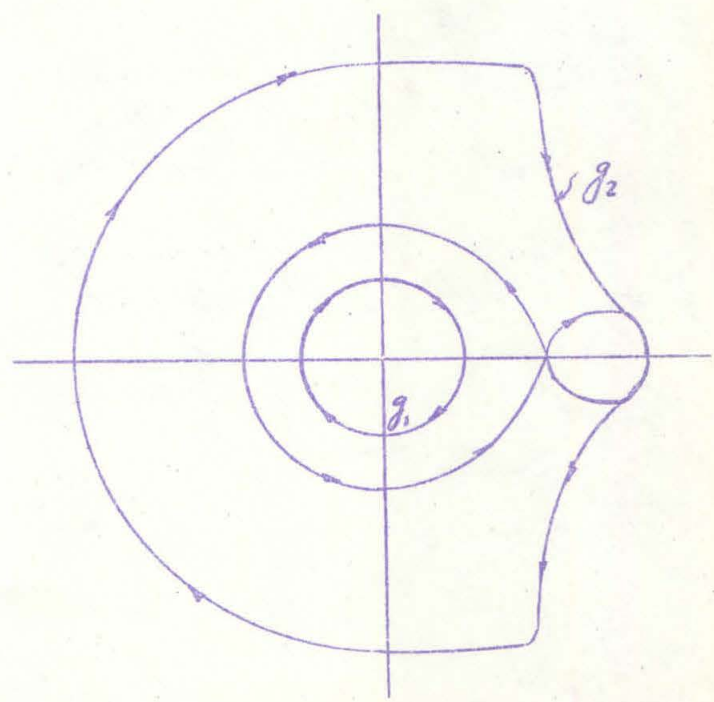
or $\overline{\Delta p}/\bar{p} > 1/14$. Stabilization for any value of δ is then possible with the chosen feed-back if $\overline{\Delta p}/\bar{p} > \frac{1}{14}$. If P = 1, $g_2(0) = -1.470$, the $g_2(p)$ contour then lies completely outside of the unit circle of $g_1(p)$. Thus we have demonstrated the feasibility of completely stabilizing the combustion by an appropriate servo-control.

Satche Diagram for General Case

In the previous section, only the simplest feed-back transfer function is considered in order not to confuse the stability criterion in the Satche diagram. If we are interested in stabilization for any value of time lag δ, then it is evident that the $g_2(p)$ contour should not cut the $g_1(p)$ contour or the unit circle. This is the primary condition for absolute stability. The g_2- contour however need not be as simple as those discussed in the previous section. For instance, the following contour is also stable:



p-CONTOUR



SATCHE DIAGRAM

It is tacitly assumed in the above discussions that the quantity within the square bracket of (10.30) has no poles or zeros on the right half of the p-plane. If this is not true, then we must first determine the difference of numbers of poles and zeros of this quantity in the right half of p-plane. This can be done easily by using the Nyquist diagram. To show this procedure clearly, let

$$\frac{pF(p)}{n}\,\frac{p+\frac{p+\frac{1}{2}}{\alpha J}}{p+p_3} = -F_1(p)$$

Or

$$F(p) = -\frac{n(p+p_3)F_1(p)}{p\left(p+\frac{p+\frac{1}{2}}{\alpha J}\right)} \qquad (10.32)$$

Therefore (10.30) becomes

$$G(p) = e^{-\delta p}[1-F_1(p)] - \left(-\frac{1}{n}\right)\frac{(p+p_1)(p+p_2)}{p+p_3} = 0 \qquad (10.33)$$

Hence if the function $1-F_1(p)$ has s poles and r zeros in the right half of p-plane, then as p traces in clockwise direction a contour enclosing the right half of p-plane, the contour $F_1(p)$ will carry out $r-s$ clockwise revolutions around the point 1.

Knowing this fact, we can divide (10.33) by $1-F_1(p)$, then the quantity $G_1(p) = G(p)/[1-F_1(p)]$ will have s zeros and r poles in addition to $G(p)$. Hence if the Satche diagram with

$$g_2(p) = -\frac{1}{n}\,\frac{(p+p_1)(p+p_2)}{(p+p_3)[1-F_1(p)]} \qquad (10.34)$$

indicates n clockwise revolutions, we know $G(p)$ will have $n+r$ zeros in the positive half plane. The stability condition is then for $g_2(p)$ given by (10.34) to lie completely outside of the unit circle and make r counterclockwise revolutions around the unit circle.

Therefore for the more general case, the stability problem can be determined by a combination of Nyquist diagram of $F_1(p)$ and Satche diagram of $g_2(p)$.

Concluding Remarks

Although the treatment presented here is for the problem of combustion instability due to time lag in a rocket chamber, it is entirely possible that the combustion instability in other power plants is due to a similar reason. If so, then such instability can be also removed by a similar feed-back control. Therefore the method given here can have other applications.

It is also conceivable that there are instances where combustion instability is desirable instead of undesirable. For example, the jet heliocopter with air compressor in the fuselage and combustion chamber at the tip of the blades. If the combustion can be made to oscillate, then the peak pressure is higher and the thermal efficiency of the motor is raised. This is sometimes called valveless pulsejet. But with proper feed-back control, the frequency and amplitude of oscillation can be controlled and gives a very flexiable system.

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Servo-Stabilization of Combustion in Rocket Motors

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This paper shows that the combustion in the rocket motor can be stabilized against any value of time lag in combustion by a feedback servo link from a chamber pressure pickup, through an appropriately designed amplifier, to a control capacitance on the propellant feed line. The technique of stability analysis is based upon a combination of the Satche diagram and the Nyquist diagram. For simplicity of calculation, only low-frequency oscillations in monopropellant rocket motors are considered. However, the concept of servo-stabilization and method of analysis are believed to be generally applicable to other cases.

THE phenomenon of rough burning in liquid-propellant rocket motor has been interpreted as the instability of the coupled system of propellant feed and combustion chamber by D. F. Gunder and D. R. Friant (1),[2] M. Yachter (2), M. Summerfield (3), and L. Crocco (4). The essential feature of these theories is the time lag between the instant of injection of the propellant and the instant when the propellant is burned into hot gas. Crocco has further improved on this concept by considering the time lag as an integrated effect of consecutive stages, each of which is controlled by the prevailing pressure in the combustion chamber. As a result of this new concept, Crocco showed the possibility of intrinsic instability with constant injection rate not influenced by the chamber pressure.

The present paper will first give a slightly more general formulation of Crocco's concept of time lag, allowing arbitrary pressure dependence of lag. Then the problem of intrinsic stability is discussed by applying a method suggested by M. Satche (5). This method is based upon a modification of the Nyquist diagram and is particularly useful for systems having time lag. For easy reference, this new diagram will be called the Satche diagram. The later sections of the paper will show the possibility of stabilizing the combustion by means of a feedback servo for all values of time lag. Such possibility of servo-stabilization was first mentioned by W. Bollay in his admirable paper (6) on the application of servomechanisms to aeronautics. The present study definitely shows the power of this idea.

Time Lag in Combustion

Let $\dot{m}_b(t)$ be the mass rate of generation of hot gas by combustion at time instant t. Consider, for simplicity, a monopropellant motor. Then the mass rate of injection at t can be denoted by $\dot{m}_i(t)$. Let $\tau(t)$ be the time lag for that parcel of propellant which is burned at the instant t. Then the mass burned during the interval from t to $t + dt$ must be equal to the mass injected during the time from $t - \tau$ to $t - \tau + d(t - \tau)$. Thus

$$\dot{m}_b(t)dt = \dot{m}_i(t - \tau)d(t - \tau) \qquad [1]$$

The mass of hot gas generated is either used to fill the combustion chamber by raising its pressure $p(t)$, or is discharged through the rocket nozzle. If the frequency of the possible oscillations in the chamber is small, then the pressure in the chamber can be considered as uniform, and as a first approximation (7) the rate of flow through the nozzle can be taken as proportional to the instantaneous chamber pressure $p(t)$. Thus if $\bar{\dot{m}}$ is the steady mass rate flow through the system, $\bar{M}_g$ is the average mass of hot gas in the chamber, and if the volume occupied by the unburned liquid propellant is neglected

$$\dot{m}_b\, dt = \bar{\dot{m}} \left(\frac{p}{\bar{p}}\right) dt + d\left(\bar{M}_g \frac{p}{\bar{p}}\right) \qquad [2]$$

where $\bar{p}$ is the steady state pressure in the combustion chamber.

By following Crocco, the nondimensional variables for the chamber pressure and the rate of injection are defined as

$$\varphi = \frac{p - \bar{p}}{\bar{p}}, \quad \mu = \frac{\dot{m}_i - \bar{\dot{m}}}{\bar{\dot{m}}} \qquad [3]$$

φ and μ are then the fractional deviation of pressure and injection rate from the average. With Equation [3], $\dot{m}_b$ can be eliminated from Equations [1] and [2], and

$$\frac{\bar{M}_g}{\bar{\dot{m}}}\frac{d\varphi}{dt} + \varphi + 1 = \left(1 - \frac{d\tau}{dt}\right)[\mu(t - \tau) + 1] \qquad [4]$$

To calculate the quantity $d\tau/dt$, Crocco's concept of pressure dependence of time lag has to be introduced. If the rate at which the liquid propellant is prepared for the final rapid transformation into hot gas is a function $f(p)$, then the lag τ is determined by

$$\int_{t-\tau}^{t} f(p)\, dt = \text{const} \qquad [5]$$

By differentiating Equation [5] with respect to t,

$$[f(p)]_t - [f(p)]_{t-\tau}\left(1 - \frac{d\tau}{dt}\right) = 0$$

The concept of small perturbation from the steady state will now be explicitly introduced: Assume that the deviation of the pressure p from the steady state value $\bar{p}$ is small. Then $f(p)$ at the instant t and $f(p)$ at the instant $t - \tau$ can be expanded as Taylor's series around $\bar{p}$. By taking only the first order terms,

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[2] Numbers in parentheses refer to the References on page 268.

$$[f(p)]_t = f(\bar{p}) + \bar{p}\left(\frac{df}{dp}\right)_{p=\bar{p}} \varphi(t)$$

$$[f(p)]_{t-\tau} = f(\bar{p}) + \bar{p}\left(\frac{df}{dp}\right)_{p=p} \varphi(t-\tau)$$

Here τ is the lag at the average pressure $\bar{p}$, a constant now. Then

$$1 - \frac{d\tau}{dt} = 1 + \left(\frac{d \log f}{d \log p}\right)_{p=\bar{p}} [\varphi(t) - \varphi(t-\tau)] \ldots [6]$$

By combining Equations [4] and [6], the following equation is obtained

$$\frac{d\varphi}{dz} + \varphi = \mu(z-\delta) + n[\varphi(z) - \varphi(z-\delta)] \ldots\ldots [7]$$

where

$$n = \left(\frac{d \log f}{d \log p}\right)_{p=\bar{p}} \ldots\ldots\ldots\ldots\ldots\ldots [8]$$

and

$$z = t/\theta_g, \quad \theta_g = \overline{M}_g/\overline{m} \ldots\ldots\ldots\ldots\ldots [9]$$

If n is a constant independent of $\bar{p}$, then $f(p)$ is proportional to p^n. This is the form of $f(p)$ assumed by Crocco. The present formulation of the problem is slightly more general in that $f(p)$ is arbitrary and the value of n is to be computed by using Equation [8], and is a function of $\bar{p}$. θ_g is, of course, the gas transit time.

Intrinsic Instability

Crocco called the instability of combustion with constant rate of injection the intrinsic instability. If the injection rate is constant and not influenced by the chamber pressure p, then $\mu \equiv 0$. Therefore the stability problem is controlled by the following simple equation obtained from Equation [7],

$$\frac{d\varphi}{dz} + (1-n)\varphi(z) + n\varphi(z-\delta) = 0 \ldots\ldots [10]$$

Now let

$$\varphi(z) \sim e^{sz}$$

Then

$$s + (1-n) + ne^{-\delta s} = 0 \ldots\ldots\ldots\ldots [11]$$

This is the equation for the exponent s.

Crocco determined the value of the complex number s by studying the set of two equations for the real and the imaginary parts of Equation [11]. However, if the point of interest is whether the system is stable or not, one can use the well-known Cauchy theorem with advantage. Let

$$G(s) = e^{-\delta s} - \left[-\frac{1-n}{n} - \frac{s}{n}\right] \ldots\ldots\ldots [12]$$

Then the question of stability is determined by whether $G(s)$ has zeros in the right half of the complex s-plane. This question itself can be in turn answered by watching the argument of $G(s)$ when s traces a contour enclosing the right half s-plane. Specifically, let s trace clockwise the contour consisting of the imaginary axis and a large half circle to the right of the imaginary axis (Fig. 1). If the vector $G(s)$ makes a number of complete clockwise

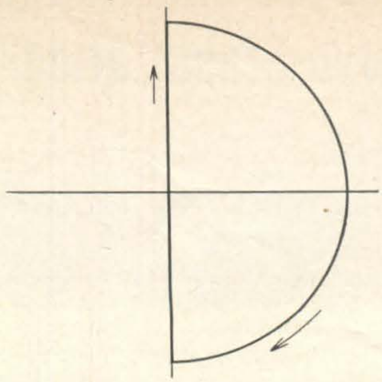

FIG. 1. CONTOUR TRACED BY THE VARIABLE s FOR THE SATCHE DIAGRAM OR THE NYQUIST DIAGRAM

revolutions, then that number is, according to Cauchy's theorem, the difference between the number of zeros and the number of poles of $G(s)$ in the right half s-plane. Since $G(s)$ evidently has no poles in the s-plane, the number of revolutions of $G(s)$ is the number of zeros. Hence for stability, the vector $G(s)$ must not make any complete revolutions, as s traces the specified contour. Therefore the stability question can be answered by plotting graphically $G(s)$ on the complex plane. This graph is, of course, the well-known Nyquist diagram.

A direct application of this method to $G(s)$ given by Equation [12] is, however, inconvenient for the complication caused by lag term $e^{-\delta s}$ (8). M. Satche (5), however, proposed a very elegant and ingenious method of treating such a system with time lag: Instead of $G(s)$, break it into two parts,

$$G(s) = g_1(s) - g_2(s) \ldots\ldots\ldots\ldots\ldots [13]$$

where

$$g_1(s) = e^{-\delta s}$$

$$g_2(s) = -\frac{1-n}{n} - \frac{s}{n} \ldots\ldots\ldots\ldots [14]$$

The vector $G(s)$ is thus a vector with vertex in $g_1(s)$ and its tail on $g_2(s)$. The graph of $g_1(s)$ is the unit circle for s on the imaginary axis. For s on the large half circle, $g_1(s)$ is within the unit circle. The graph of $g_2(s)$ is the straight line (Fig. 2) paralleled to the imaginary axis when s is on the imaginary axis. When s is on the large half circle, $g_2(s)$ is a half of a large circle closing the contour on the left. A moment's reflection will show that in order for the vector $G(s)$ not to make complete revolutions for any value of δ, the

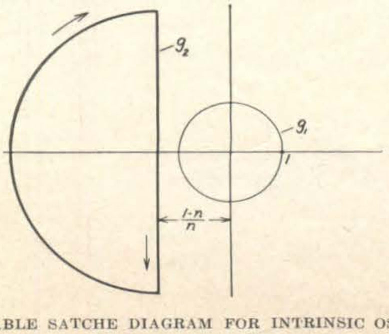

FIG. 2. STABLE SATCHE DIAGRAM FOR INTRINSIC OSCILLATIONS; $0 < n < 1/2$

$g_2(s)$ contour must lie *completely* out of the $g_1(s)$ contour. That is, for unconditional intrinsic stability

$$\frac{1-n}{n} > 1 \quad \text{or} \quad \frac{1}{2} > n > 0 \qquad [15]$$

When $n > {}^1/_2$, the $g_1(s)$ contour and the $g_2(s)$ contour intersect. Stability is still possible, however, if for $g_2(s)$

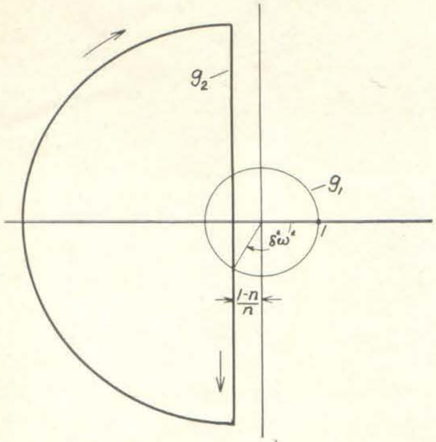

FIG. 3. UNSTABLE SATCHE DIAGRAM FOR INTRINSIC OSCILLATIONS; $n > {}^1/_2$

within the unit circle (Fig. 3), $g_1(s)$ is to the right of $g_2(s)$. This condition is satisfied if

$$\cos(\delta\sqrt{2n-1}) > -\frac{1-n}{n}$$

Or if

$$\delta < \delta^*$$

where

$$\delta^* = \frac{1}{\sqrt{2n-1}}\cos^{-1}\left(-\frac{1-n}{n}\right) = \frac{1}{\sqrt{2n-1}}\left(\pi - \cos^{-1}\frac{1-n}{n}\right) \qquad [16]$$

When $\delta = \delta^*$, then with

$$\omega^* = \sqrt{2n-1} \qquad [17]$$

$G(i\omega^*) = 0$. Therefore when $\delta = \delta^*$, φ has the oscillatory solution with the angular frequency ω^*.

These results on intrinsic stability were obtained by Crocco. The present discussion with the Satche diagram, however, seems to be simpler. For the more complicated stability problem treated below with feed system and servo control, the solution is hardly practical without the Satche diagram.

System Dynamics with Servo Control

Consider now a system including the propellant feed and a servo control represented by Fig. 4. In order to approximate the elasticity of the feed line, a spring load capacitance is put at the midway point between the propellant pump and the injector. The spring constant is to be computed from the feed-line dimensions.[3] Near the injector there is another capacitance controlled by the servo. The servo receives its signal from the chamber pressure pickup through an amplifier. If the feed system and the motor design are fixed by the designer, the question is whether it is possible to design an appropriate amplifier so that the whole system will be stable. Because there is no accurate information on the time lag of combustion, a practical design should specify unconditional stability, i.e., stability for any value of δ.

Let $\dot{m}_0$ be the instantaneous mass flow rate out of the propellant pump, and p_0 be the instantaneous pressure at the outlet of pump. The average flow rate must be $\bar{\dot{m}}$. The average pressure is $\bar{p}_0$. The pump characteristics can be represented by the following equation,

$$\frac{p_0 - \bar{p}_0}{\bar{p}_0} = -\alpha\frac{\dot{m}_0 - \bar{\dot{m}}}{\dot{m}} \qquad [18]$$

If the time rate of change of mass flow is small, α is simply related to the slope of the head-volume curve of the pump at constant speed near the steady-state operating point. For constant pressure pump or the simple pressure feed, α is zero. For conventional centrifugal pumps, α is approximately 1. For displacement pumps, α is very large.

Let $\dot{m}_1$ be the instantaneous mass rate of flow after the spring loaded capacitance, χ the spring constant of the capacitance, and p_1 the instantaneous pressure at the capacitance. Then

$$\dot{m}_0 - \dot{m}_1 = \rho\chi\frac{dp_1}{dt} \qquad [19]$$

where ρ is the density of the propellant, a constant.

In the following calculation, the pressure drop in the line by frictional forces will be neglected. Then the pressure difference $p_0 - p_1$ is due to the acceleration of the flow only. That is

$$p_0 - p_1 = \frac{l}{2A}\frac{d\dot{m}_0}{dt} \qquad [20]$$

where A is the cross-sectional area of the feed line, a constant, and l is the total length of the feed line. Similarly, if p_2 is the instantaneous pressure at the control capacitance.

$$p_1 - p_2 = \frac{l}{2A}\frac{d\dot{m}_1}{dt} \qquad [21]$$

If the mass capacity of the control capacitance is C, then

$$\dot{m}_1 - \dot{m}_i = \frac{dC}{dt} \qquad [22]$$

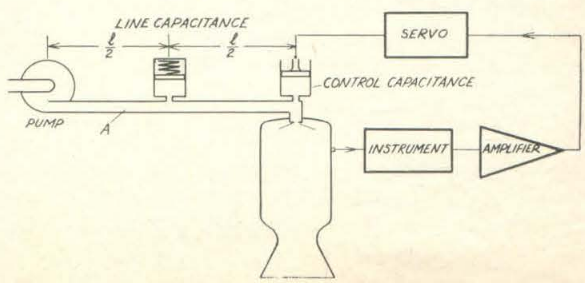

FIG. 4. SERVO-CONTROLLED LIQUID MONOPROPELLANT ROCKET MOTOR

[3] See the Appendix for details.

Since the control capacitance is very close to the injector, the inertia of the mass of propellant between the control capacitance and the injector is negligible. Then

$$p_2 - p = \frac{1}{2}\frac{\dot{m}_i^2}{\rho A_i^2} \qquad [23]$$

where A_i is the effective orifice area of the injector. A_i can be eliminated from the calculation by noting that at steady state, the difference of pressures $\bar{p}_0$ and $\bar{p}$, or $\Delta\bar{p}$ is

$$\bar{p}_0 - \bar{p} = \Delta\bar{p} = \frac{1}{2}\frac{\bar{\dot{m}}^2}{\rho A_i^2} \qquad [24]$$

Equations [18] to [24] describe the dynamics of the feed system. By a straightforward process of elimination of variables, a relation between $\dot{m}_i$, p, and C is obtained. To express this relation in nondimensional form, the following quantities are introduced, following the notation of Crocco:

$$P = \frac{\bar{p}}{2\Delta\bar{p}},\quad E = \frac{2\Delta\bar{p}}{\bar{\dot{m}}\theta_g}\rho\chi,\quad J = \frac{l\bar{\dot{m}}}{2\Delta\bar{p}A\theta_g} \qquad [25]$$

and

$$\kappa = C/\bar{\dot{m}}\theta_g \qquad [26]$$

where θ_g is the gas transit time given by Equation [9]. Then the nondimensional equation relating φ, μ, and κ is

$$P\left\{1 + E\left(P+\frac{1}{2}\right)\frac{d}{dz} + \frac{JE}{2}\frac{d^2}{dz^2}\right\}\varphi + \left[\left\{1+\alpha\left(P+\frac{1}{2}\right)\right\} + \left\{\alpha E\left(P+\frac{1}{2}\right)+J\right\}\frac{d}{dz} + \left\{\frac{\alpha JE}{2}\left(P+\frac{1}{2}\right)+\frac{JE}{2}\right\}\frac{d^2}{dz^2} + \frac{J^2E}{4}\frac{d^3}{dz^3}\right]\mu + \left[\alpha\left(P+\frac{1}{2}\right)\frac{d}{dz} + J\frac{d^2}{dz^2} + \frac{\alpha JE}{2}\left(P+\frac{1}{2}\right)\frac{d^3}{dz^3} + \frac{J^2E}{4}\frac{d^4}{dz^4}\right]\kappa = 0 \qquad [27]$$

where z is the nondimensional time variable defined by Equation [9].

The dynamics of the servo control are specified by the composite of the instrument characteristics of the pressure pickup, the response of the amplifier, and the properties of the servo. Since it is not the purpose of the present paper to discuss the detailed design of the servo control, the over-all dynamics of the servo control are represented by the following operator equation:

$$F\left(\frac{d}{dz}\right)\varphi = \kappa \qquad [28]$$

where F is the ratio of two polynomials with the denominator of higher order than the numerator.

Equations [7], [27], and [28] are the three equations for the three variables φ, μ, and κ. Since they are equations with constant coefficients, the appropriate forms for the variables are

$$\varphi = ae^{sz},\quad \mu = be^{sz},\quad \kappa = ce^{sz} \qquad [29]$$

By substituting Equation [29] into Equations [7,] [27], and [28], three homogeneous equations for a, b, and c are obtained. In order for a, b, c to be nonzero, the determinant formed by their coefficients must vanish. This condition can be written as follows:

$$[s+(1-n)]\left[\frac{J^2E}{4}s^3 + \frac{JE}{2}\left\{1+\alpha\left(P+\frac{1}{2}\right)\right\}s^2 + \left\{\alpha E\left(P+\frac{1}{2}\right)+J\right\}s + \left\{1+\alpha\left(P+\frac{1}{2}\right)\right\}\right] + e^{-\delta s}\left\{\frac{nJ^2E}{4}s^3 + \left[\frac{nJE}{2}\left\{1+\alpha\left(P+\frac{1}{2}\right)\right\} + \frac{JEP}{2}\right]s^2 + \left[n\left\{\alpha E\left(P+\frac{1}{2}\right)+J\right\} + \alpha EP\left(P+\frac{1}{2}\right)\right]s + \left[n\left\{n+\alpha\left(P+\frac{1}{2}\right)\right\}+P\right] + sF(s)\left[\frac{J^2E}{4}s^3 + \frac{\alpha JE}{2}\left(P+\frac{1}{2}\right)s^2 + Js + \alpha\left(P+\frac{1}{2}\right)\right]\right\} = 0 \qquad [30]$$

This is the equation for determining the exponent s. $F(s)$ is now recognized as the over-all transfer function of the servo-control link. The complete system stability depends upon whether Equation [30] gives roots that have positive real parts.

Instability Without Servo Control

The system characteristics without the servo control can be simply obtained from the basic Equation [30] by setting $F(s) = 0$. Let it be assumed that the polynomial multiplied into $e^{-\delta s}$ has no zero in the positive half s-plane, as is usually the case. Then Equation [30] can be divided by that polynomial without introducing poles in the positive half s-plane into the resultant function. That is, for the Satche diagram, one has again

$$G(s) = g_1(s) - g_2(s),\qquad g_1(s) = e^{-\delta s}$$

$g_1(s)$ is thus again the "unit circle." $g_2(s)$ is now much more complicated:

$$g_2(s) = -\left[\frac{s}{n}+\frac{1-n}{n}\right]\frac{\frac{J^2E}{4}s^3 + \frac{JE}{2}\left\{1+\alpha\left(P+\frac{1}{2}\right)\right\}s^2 + \left\{\alpha E\left(P+\frac{1}{2}\right)+J\right\}s + \left\{1+\alpha\left(P+\frac{1}{2}\right)\right\}}{\frac{J^2E}{4}s^3 + \frac{JE}{2}\left\{1+\alpha\left(P+\frac{1}{2}\right)+\frac{P}{n}\right\}s^2 + \left\{\alpha E\left(P+\frac{1}{2}\right)\left(1+\frac{P}{n}\right)+J\right\}s + \left\{1+\alpha\left(P+\frac{1}{2}\right)+\frac{P}{n}\right\}} \qquad [31]$$

The intercept of $g_2(s)$, when s is pure imaginary, is given by setting $s = 0$ in Equation [31], i.e.,

$$g_2(0) = -\frac{1-n}{n}\,\frac{1+\alpha\left(P+\frac{1}{2}\right)}{1+\alpha\left(P+\frac{1}{2}\right)+\frac{P}{n}} \qquad [32]$$

Since all the parameters n, α, P are positive, the magnitude of $g_2(0)$ is now smaller than the magnitude of $g_2(0)$ given by Equation [14] for the intrinsic stability problem. Thus the effect of the feed system is to move the $g_2(s)$ curve toward the unit circle of $g_1(s)$ in the Satche diagram. For instance, for $n = {}^1/_2$, $g_2(s)$ is just tangent to the unit circle for the intrinsic system without considering the propellant feed. But with the propellant feed system, $g_2(s)$ contour will intersect the unit circle and the system will become unstable, for time lag δ exceeds a certain finite value. The influence of the feed system is thus always destabilizing. This is further confirmed by considering the asymptote of $g_2(s)$ for large imaginary s, obtained from Equation [31]. That is

$$g_2(s) \sim -\left[\frac{s}{n} + \left(\frac{1-n}{n} - \frac{2P}{Jn^2}\right) + \ldots\right], \qquad |s| \gg 1 \ldots [33]$$

Therefore, for large imaginary s, $g_2(s)$ approaches asymptotically a line parallel to the imaginary axis at a distance

$$\frac{1-n}{n} - \frac{2P}{Jn^2}$$

to the left of the imaginary axis. The effect of the feed system is again to move $g_2(s)$ toward the unit circle.

It is thus evident that for the parameter n near $^1/_2$ or larger than $^1/_2$, it would be impossible to design the system for unconditional stability. In the Satche diagram, $g_1(s)$ contour and $g_2(s)$ will always intersect without a servo control.

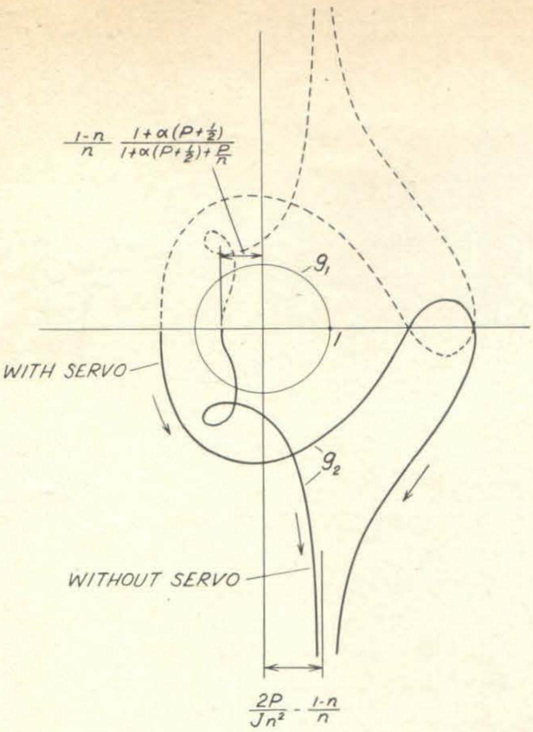

FIG. 5. SATCHE DIAGRAM FOR THE ORIGINAL AND FOR THE SERVO-STABILIZED SYSTEM

Complete Stability with Servo Control

If the polynomial $H(s)$

$$H(s) = \frac{J^2E}{4}s^3 + \left[\frac{JE}{2}\left\{1 + \alpha\left(P + \frac{1}{2}\right)\right\} + \frac{JEP}{2n}\right]s^2 + \left[\alpha E\left(P + \frac{1}{2}\right) + \frac{\alpha EP}{n}\left(P + \frac{1}{2}\right)\right]s + \left[1 + \alpha\left(P + \frac{1}{2}\right) + \frac{P}{n}\right] + \frac{1}{n}sF(s)\left[\frac{J^2E}{4}s^3 + \frac{\alpha JE}{2}\left(P + \frac{1}{2}\right)s^2 + Js + \alpha\left(P + \frac{1}{2}\right)\right] \ldots [34]$$

which multiplies into $e^{-\delta s}$ in Equation [30], has no poles and zeros in the right half s-plane, then the occurrence zeros of the expression in Equation [30] in the right half s-plane can be determined from the Satche diagram with

$$g_1(s) = e^{-\delta s}$$

and

$$g_2(s) = -\left[\frac{s}{n} + \frac{1-n}{n}\right]\left[\frac{J^2E}{4}s^3 + \frac{JE}{2}\left\{1 + \alpha\left(P + \frac{1}{2}\right)\right\}s^2 + \left\{\alpha E\left(P + \frac{1}{2}\right) + J\right\}s + \left\{1 + \alpha\left(P + \frac{1}{2}\right)\right\}\right] \Big/ H(s) \ldots [35]$$

As s traces the contour of Fig. 1, $g_1(s)$ is again a unit circle. Therefore, if simultaneously the $g_2(s)$ contour is completely outside the unit circle, there can be no root of Equation [30] in the right half s-plane. In other words, if the transfer function $F(s)$ of the servo-control link is so designed as to place the $g_2(s)$ contour completely out of the unit circle (Fig. 5), then the system is stabilized for all time lags.

As an example, take

$$n = \frac{1}{2}, \quad P = \frac{3}{2}, \quad J = 4, E = \frac{1}{4}, \quad \alpha = 1$$

Then without the servo control, the $g_2(s)$ is

$$g_2(s) = -\frac{1}{2}\frac{(2s+1)(2s^3 + 3s^2 + 9s + 6)}{s^3 + 3s^2 + 6s + 6}$$

Of primary interest is the behavior of $g_2(s)$ when s is a pure imaginary number $i\omega$, ω real. Thus

$$g_2(i\omega) = -\frac{1}{2}\frac{(6 - 21\omega^2 + 4\omega^4)(6 - 3\omega^2) + \omega^2(21 - 8\omega^2)(6 - \omega^2)}{(6 - 3\omega^2)^2 + \omega^2(6 - \omega^2)^2} - \frac{1}{2}i\omega\frac{(21 - 8\omega^2)(6 - 3\omega^2) - (6 - 21\omega^2 + 4\omega^4)(6 - \omega^2)}{(6 - 3\omega^2)^2 + \omega^2(6 - \omega^2)^2}$$

This contour for $\omega \geq 0$ is plotted in Fig. 6. It is evident that for sufficiently large values of time lag, the system will be unstable. On the other hand, if the $g_2(s)$ contour can be changed by the servo control to, say,

$$g_2(s) = -2\frac{(s+2)(s+3)}{(s+6)}$$

Then, as plotted in Fig. 6, the new g_2 contour is completely outside of the unit circle of $g_1(s)$. Therefore the system is now unconditionally stable. A straightforward calculation from Equations [31] and [35] shows that the required transfer function $F(s)$ for the servo link is

$$F(s) = -4.875\frac{(s + 1.0528)(s^2 + 0.7164s + 2.6304)}{s(s+2)(s+3)(s+0.5332)(s^2 + 0.4668s + 3.7511)}$$

The servo link has thus the character of an integrating circuit. If, with given response of the chamber pressure pickup and of the servo for the control capacitance, an amplifier could be designed to give an over-all transfer function close to that specified above, the combustion can be stabilized by such a servo control.

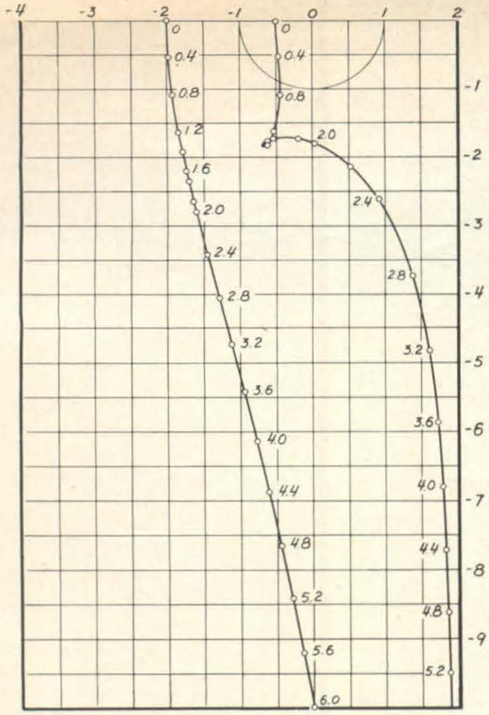

FIG. 6. SATCHE DIAGRAM FOR THE ORIGINAL AND FOR THE SERVO-STABILIZED SYSTEM

$P = {}^3/_2,\ J = 4,\ E = {}^1/_4,\ \alpha = 1$

($g_2(i\omega)$ without servo intersects the unit circle; $g_2(i\omega)$ with servo is outside the unit circle. Numbers beside points are the value of ω.)

As the second example, take

$$n = \frac{1}{2},\ P = \frac{3}{2},\ J = 4,\ E = \frac{1}{4},\ \alpha = 0$$

Since $\alpha = 0$, the feed pressure p_0 is thus constant with even variable flow of propellant. The case then corresponds to that of a simple pressure feed. Without the servo control,

$$g_2(s) = -\frac{1}{2}\frac{(2s + 1)(2s^3 + s^2 + 8s + 2)}{s^3 + 2s^2 + 4s + 4}$$

When s is pure imaginary,

$$g_2(i\omega) = -\frac{1}{2}\frac{(4 - 2\omega^2)(2 - 17\omega^2 + 4\omega^4) + \omega^2(4 - \omega^2)(12 - 4\omega^2)}{(4 - 2\omega^2)^2 + \omega^2(4 - \omega^2)^2} - \frac{1}{2}i\omega\frac{(4 - 2\omega^2)(12 - 4\omega^2) - (4 - \omega^2)(2 - 17\omega^2 + 4\omega^4)}{(4 - 2\omega^2)^2 + \omega^2(4 - \omega^2)^2}$$

This contour of g_2 is plotted in Fig. 7. It is evident that without servo control the combustion will be unstable for sufficiently long time lag. In fact, the system is even less stable than the system considered in the first example: It will become unstable at shorter time lag. The part of the g_2 contour near $\omega = 2$ is of special interest. Near $\omega = 2$, the contour comes so close to the unit circle of g_1 that if the value of time lag δ is such as to make g_1 and g_2 for $\omega \sim 2$ very close to each other, then an almost undamped oscillation at $\omega \sim 2$ can occur. This critical value of δ is evidently smaller than the critical δ determined from the true intersection of g_2 with the unit circle at $\omega \sim 0.65$. Such near instability at smaller values of time lag can be easily overlooked in the analytic treatment of the stability condition by Crocco, and yet such possible instability should not be dismissed. This, perhaps, indicates the superiority of the present graphical method.

For unconditional stability, g_2 should be displaced out of the unit circle, to, say, the same "stable" contour as in the first example. The required transfer function $F(s)$ is calculated to be

$$F(s) = -4.875\frac{(s + 0.8126)(s^2 - 0.04337s + 2.6506)}{s^2(s + 2)(s + 3)(s^2 + 4)}$$

The required servo link must then have the character of double integrating circuit. Furthermore, the transfer function has two purely imaginary poles at $\pm 2i$. This unrealistic requirement on the amplifier comes from the original feed-system dynamics and is due to the neglect of frictional damping in the feed line. In any actual system, the frictional damping in the feed line will remove these purely imaginary poles of the required transfer function $F(s)$ and replace them by two complex conjugate poles.

Stability Criteria

In the preceding discussion of servo-stabilization, it is assumed that the polynomial $H(s)$, Equation [34], has no pole or zero in the right half s-plane. This is, however, not necessarily the case. In general then, one should first investigate the number of zeros and poles of $H(s)$ in the right half s-plane. To do this, it

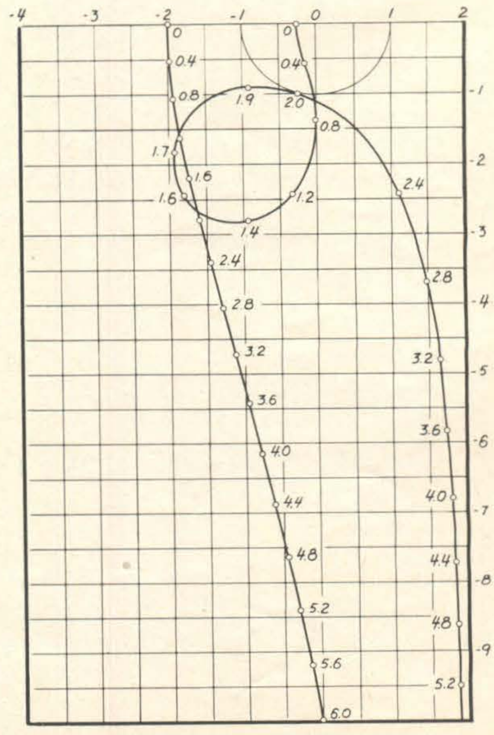

FIG. 7. SATCHE DIAGRAM FOR THE ORIGINAL AND FOR THE SERVO-STABILIZED SYSTEM

$P = {}^3/_2,\ J = 4,\ E = {}^1/_4,\ \alpha = 0$

($g_2(i\omega)$ without servo intersects the unit circle; $g_2(i\omega)$ with servo is outside the unit circle. Numbers beside points are the value of ω.)

should be recognized that the polynomial in Equation [34] before the factor $F(s)$ usually does not have zeros in the right half s-plane. Therefore instead of studying $H(s)$, one can study the ratio of $H(s)$ and that polynomial. That is, the number of zeros and poles of $H(s)$ in the right half s-plane is the same as the number of zeros and poles of the following function

$$1 + K(s) = \frac{H(s)}{\frac{J^2E}{4}\, s^3 + \left[\frac{JE}{2}\left\{1 + \alpha\left(P + \frac{1}{2}\right)\right\} + \frac{JEP}{2n}\right] s^2 + \left[\alpha E\left(P + \frac{1}{2}\right) + \frac{\alpha EP}{n}\left(P + \frac{1}{2}\right)\right] s + \left[1 + \alpha\left(P + \frac{1}{2}\right) + \frac{P}{n}\right]} \quad \ldots [36]$$

where

$$K(s) = \frac{\frac{1}{n}\, sF(s)\left[\frac{J^2E}{4}\, s^3 + \frac{\alpha JE}{2}\left(P + \frac{1}{2}\right) s^2 + Js + \alpha\left(P + \frac{1}{2}\right)\right]}{\frac{J^2E}{4}\, s^3 + \left[\frac{JE}{2}\left\{1 + \alpha\left(P + \frac{1}{2}\right)\right\} + \frac{JEP}{2n}\right] s^2 + \left[\alpha E\left(P + \frac{1}{2}\right) + \frac{\alpha EP}{n}\left(P + \frac{1}{2}\right)\right] s + \left[1 + \alpha\left(P + \frac{1}{2}\right) + \frac{P}{n}\right]} \quad \ldots [37]$$

According to the Nyquist criterion, the number of poles and zeros for $1 + K(s)$ in the right half s-plane can be found by plotting the Nyquist diagram of $1 + K(s)$ with s tracing the contour of Fig. 1. In fact, if $1 + K(s)$ or $H(s)$ has r zeros and q poles in right half s-plane then $K(s)$ will carry out $r - q$ clockwise revolutions around the point -1, as s traces the contour of Fig. 1. Hence the necessary information on $H(s)$ can be obtained by plotting the Nyquist diagram of $K(s)$.

When one divides the Equation [30] by $H(s)$ in order to obtain $g_1(s)$ and $g_2(s)$ as given by Equation [35], q zeros and r poles are introduced in the right half s-plane. The q poles of $K(s)$ must come from $F(s)$, since the polynomial in the denominator of Equation [37] has no zero in the right half s-plane. Therefore the original expression in Equation [30] also has q poles in the right half s-plane. Hence in order for the original expression in Equation [30] to have no zero in the right half s-plane, $g_2(s)$ must make $-q + (q - r) = -r$ clockwise revolutions around the unit circle. In order for stability to be unconditional, i.e., stable for all time lag, the $g_2(s)$ contour should never intersect the unit circle. Therefore the general unconditional stability criteria are, first, $g_2(s)$ contour completely outside of the unit circle; and, second, $g_2(s)$ making r counterclockwise revolutions around the unit circle as s traces the conventional contour enclosing the right half s-plane. These are the criteria for stability with the Satche diagram. To determine r, one has to use the Nyquist diagram of $K(s)$, Equation [37]. Thus the stability problem for the general case requires both the Satche diagram and the Nyquist diagram (Fig. 8).

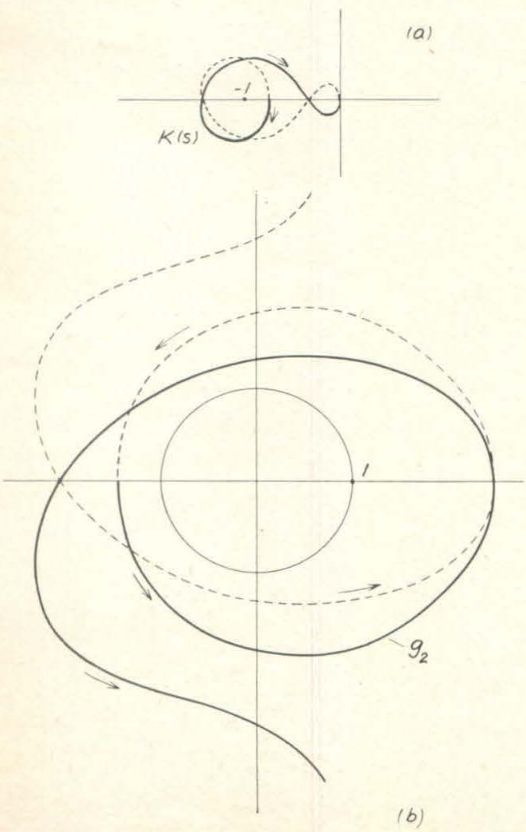

FIG. 8. FULL CURVE FOR POSITIVE ω; DOTTED CURVE FOR NEGATIVE ω.

(a) Nyquist diagram for $K(s)$, with two zeros for $1 + K(s)$ in right half s-plane. (b) Corresponding stable Satche diagram.

Concluding Remarks

In the previous sections of this paper, the theoretical possibility of completely stabilizing the combustion for any value of time lag by servo control is demonstrated. The great flexibility of electronic amplifier seems to indicate that this theoretical possibility can be always realized. On the other hand, without the servo link, unconditional stability is shown to be generally impossible. Therefore the concept of feedback servo is indeed a powerful tool in controlling the behavior of a time-lag system. It is to be realized, of course, that the proposed scheme is but one among many. No attempt is made here to give an exhaustive treatment of all possible schemes. The best scheme is certainly to be determined by detailed considerations on all aspects of the engineering problem, such as the possibility of high-frequency acoustic oscillations which are not considered here. The main purpose here is to give a general discussion of the concept together with a suggested general method of analyzing the stability by the Satche diagram.

(Continued on page 268)

It is of interest to point out that stabilization by servo control is only one phase of the general concept of feedback link. The opposite case of destabilization could be of importance also. For instance, consider the so-called valveless pulsejet. It is not always possible to operate the engine with the desired pulsation. With a feedback servo linking the combustion chamber pressure pickup through an amplifier to the fuel line, the system can be destabilized at the desired operating frequency and thus operate the engine at that frequency of pulsation. This application of servo-destabilization gives the valveless pulsejet a new flexibility and an extended range of operation. Therefore it seems worth while to explore carefully all possible applications of feedback control to systems with time lag.

APPENDIX

Calculation of Parameters J and E

If L^* and c^* are the characteristic length and the characteristic velocity of the motor, and if T_c is the chamber temperature, R the gas constant, the transit time θ_g is $\theta_g = L^*c^*/RT_c$.

To calculate J and E defined by Equation [25], it is more convenient to use the average propellant velocity v in the feed line. Thus $\bar{m} = \rho A v$.

Thus, according to Equation [25]

$$J = \frac{1}{2}\rho v \left(\frac{l}{\theta_g}\right) \Big/ \Delta\bar{p}$$

A consistent set of units would be ρ in slugs per cubic foot, v in feet per second, l in feet, θ_g in seconds, and $\Delta\bar{p}$ in pounds per square foot.

If d is the diameter of the feed line, h its thickness, and E' Young's modulus of the tube material, then χ, the change in volume of the feed line per unit rise in pressure, is

$$\chi = l\pi \left(\frac{d}{2}\right)^2 d \Big/ E'h$$

Therefore Equation [25] gives

$$E = \frac{2\Delta\bar{p}}{E'}\left(\frac{d}{h}\right)\frac{l/\theta_g}{v}$$

A consistent set of units would be $\Delta\bar{p}$ in pounds per square inch, E' in pounds per square inch, l in feet, θ_g in seconds, and v in feet per second.

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4 "Aspects of Combustion Stability in Liquid Propellant Rocket Motors, Parts I and II," by L. Crocco, Journal of the American Rocket Society, vol. 21, November 1951, pp. 163–178; vol. 22, January-February 1952, pp. 7–16.

5 Discussion by M. Satche on "Stability of Linear Oscillating Systems with Constant Time Lag," by H. I. Ansoff, *Journal of Applied Mechanics*, vol. 16, December 1949, pp. 419–420.

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7 "The Transfer Functions of Rocket Nozzles," by H. S. Tsien, Journal of the American Rocket Society, vol. 22, May-June 1952, pp. 139–143.

8 See, for instance, "Stability of Linear Oscillating Systems with Constant Time Lag," by H. I. Ansoff, *Journal of Applied Mechanics*, vol. 16, June 1949, pp. 158–164.

11. Stability Problem with Time-Varying Coefficients - Artillery Rockets

In the previous chapters, the systems for which stability and control were discussed, were all systems characterized by differential equations with constant coefficients. Mathematically, such systems are the simplest systems where the method of Laplace transform is most effective in dealing with their transient solutions under initial disturbances. As pointed out in chapter 6, the problems of next order of mathematical difficulty is the stability problem for systems characterized by time-varying coefficients. Such systems are still linear but the method of Laplace transform is not applicable. In this chapter, we shall only consider a simplest problem of this type: Namely, the problem of motion of artillery rocket during burning. The general problem of artillery rocket has been treated by various authors, mostly during the World War II. The American work is summarized by J. B. Rosser, R. R. Newton and G. L. Gross.* The British work is summarized by R. A. Rankin.** A French paper is that of P. Carriere.*** The following account follows that of Rosser and collaborators, for spinless rockets.

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* J. B. Rosser, R. R. Newton, G. L. Gross, "Mathematical Theory of Rocket Flight" McGraw Hill, N.Y. (1947).

** R. A. Rankin, "The Mathematical Theory of the Motion of Rotated and Unrotated Rockets" Phil. Trans. Royal Society of London, (A), 241: 457-585 (1949).

*** P. Carriere "Perturbations ballistiques d'un projectile autopropulsé à poudre, pendant la phase d'autopropulson". Mémorial de l'Artillerie Francaise, Vol. 25, pp. 253-360 (1951).

Jet Damping

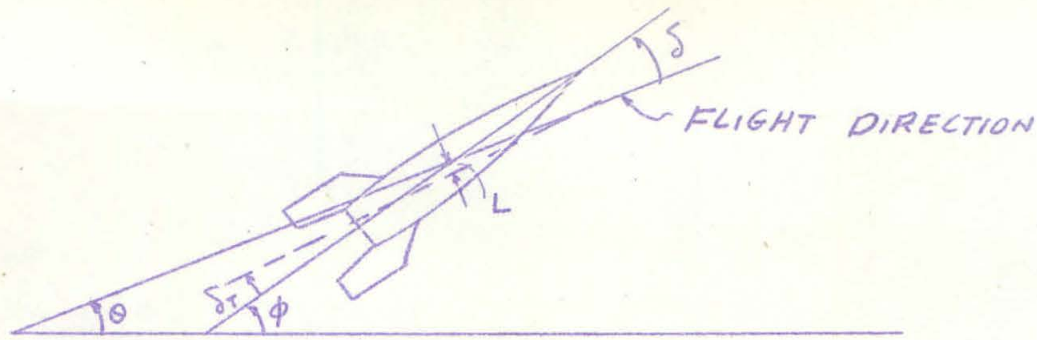
The difference between a shell and a rocket is only due to the presence of the exhaust jet. It is well-known that the exhaust jet gives a propulsive force T calculated by

$$T = \dot{m} c \tag{11.1}$$

where $\dot{m}$ is the mass rate of flow in the jet and c the effective exhaust velocity. When the rocket has a yawing and pitching motion, the exhaust jet gives rise to another important quantity, the jet damping moment. To calculate this quantity, one needs the help of a theorem which can be derived from the principles of mechanics. This theorem is:

"If one has a system of S particles, then the moment of the exterior forces acting on S , taken with respect to any axis which passes through the center of gravity of S and is fixed in direction, is equal to the time rate of change of the moment of momentum of S , taken with respect to that axis, plus the rate at which the particles that are leaving S are transferring moment of momentum, taken with respect to that axis, out of S".

The mass rate of flow out of the rocket is $\dot{m}$. Now if we assume that the yawing and pitching oscillation of the rocket is slow enough for the flow in the nozzle to be quasi-steady, then the gas always leaves the nozzle with a relative velocity in line with its axis if the nozzle is perfect; if not, at a fixed angle with respect to the nozzle axis. Then the side-wise velocity of the exhaust gas is caused only by the yawing and pitching motion of the rocket. Let the yawing or the pitching angle be ϕ , and r_e be the distance between the center of gravity and the exit of nozzle, then the side-wise velocity is $r_e \dot{\phi}$. The momentum is $\dot{m} r_e \dot{\phi}$.



The moment of momentum is thus $\dot{m} r_e^2 \dot{\phi}$. Let the instantaneous mass of the rocket be M, the radius of gyration be k , the aerodynamic stalling moment be $-J_a$, the other stalling moment be $-J$. Then the application of the above stated theorem gives

$$\frac{d}{dt}(Mk^2\dot{\phi}) + \dot{m} r_e^2 \dot{\phi} = -J_a - J \tag{11.2}$$

But

$$\frac{d}{dt}(Mk^2\dot{\phi}) = -\dot{m} k^2 \dot{\phi} + M\dot{\phi}\frac{dk^2}{dt} + Mk^2\ddot{\phi}$$

The term dk^2/dt is however quite small, and so it can be neglected. Then (11.2) becomes

$$Mk^2\ddot{\phi} = -\dot{m}\dot{\phi}(r_e^2 - k^2) - J_a - J \tag{11.3}$$

The left side is now the conventional product of moment of inertia and the angular acceleration. The first term on the right side is the moment due to the exhaust jet and its sense is such as to decrease the angular velocity $\dot{\phi}$. Therefore it is called the jet damping moment.

Equations of Motion

The ballisticians define their aerodynamic coefficient in somewhat different way than the aeronautical engineers. The drag force is

$$K_D \rho d^2 v^2$$

where ρ the air density, d the reference diameter, and v the flight velocity. The more familiar definition is

$$C_D \frac{\rho}{2} v^2 \frac{\pi}{4} d^2$$

Therefore $K_D = C_D \frac{\pi}{8}$

Now if θ is the inclination of flight path, δ the angle of attack, δ_T the deviation of the thrust axis from the axis of the rocket, the equation in the vertical plane is then

$$M\dot{v} = T\cos(\delta - \delta_T) - Mg\sin\theta - K_D \rho d^2 v^2 - F_1 \qquad (11.4)$$

where F_1 is any other forces such as the friction with the launcher, etc. A similar equation can be written for the motion in the horizontal plane.

For the angle of inclination, if δ_L is the angle of zero lift, which will zero if perfect symmetry is attained in manufacturing process, and if K_L is the "lift coefficient" and K_S is the coefficient of lift due to yawing or pitching velocity, then

$$M v\dot{\theta} = T\sin(\delta - \delta_T) - Mg\cos\theta + K_L \rho d^2 v^2 \sin(\delta - \delta_L) + K_S \rho d^3 v \dot{\phi} \cos\delta - F_2 \qquad (11.5)$$

in the vertical plane and

$$M v\dot{\theta} = T\sin(\delta - \delta_T) + K_L \rho d^2 v^2 \sin(\delta - \delta_L) + K_S \rho d^3 v \dot{\phi} \cos\delta - F_2 \qquad (11.6)$$

in the horizontal plane. F_2 is other force components normal to the flight path. In (11.5) and (11.6) we have written down the same δ_T and δ_L for the equations in vertical plane and horizontal plane. Actually the numerical values for them are different for the two planes, and this fact should be kept in mind.

If L is the moment arm of the thrust due to the misalignment of the thrust axis, K_M the coefficient for moment about the center of gravity due to angle of attack, K_H the coefficient for moment due to angular velocity, and δ_M the angle of zero moment, then the equation for angular motion of the rocket is

$$Mk^2\ddot{\phi} = TL - K_M \rho d^3 v^2 \sin(\delta - \delta_M) - K_H \rho d^4 v \dot{\phi} - \dot{m}\dot{\phi}(r_e^2 - k^2) - J \qquad (11.7)$$

where J is the other moment about the center of gravity due to reactions from the launcher, etc.

Simplification of the Basic Equations

The systems of equations (11.4), (11.5), (11.6) and (11.7) can be greatly simplified if we make clear the purpose of our analysis. Our purpose is to calculate the disturbed motion of the rocket during the burning of the rocket. Of the disturbed quantities, we are particularly interested in the disturbance of θ and δ. Then the velocity v need be only calculated up to the zeroth order, or the undisturbed case. During burning, the thrust predominates all other forces, thus for zeroth order v, (11.4) simplifies to

$$M\dot{v} = T \qquad (11.8)$$

After burning, the gravitational and the aerodynamic drag must be considered. Therefore a more general form of the equation for zeroth order in v is

$$M\dot{v} = T - Mg\sin\theta_a - K_D \rho d^2 v^2 - F_1 \qquad (11.9)$$

where θ_a is the average angle of the inclination of flight path. The deviation is considered to be small. Similarly (11.5) and (11.6) can be written as the following, taking only first order terms,

$$Mv\dot{\theta} = T(\delta-\delta_T) - Mg\cos\theta_a + K_L\rho d^2v^2(\delta-\delta_L) + K_S\rho d^3v\dot{\phi} - F_2$$

for vertical plane and

$$Mv\dot{\theta} = T(\delta-\delta_T) + K_L\rho d^2v^2(\delta-\delta_L) + K_S\rho d^3v\dot{\phi} - F_2$$

for the horizontal plane. However, if we can amalgamate these two equations by introducing complex variables δ, ϕ and θ whose real parts are the values of δ, ϕ and θ in the vertical plane and whose imaginary parts are the values of δ, ϕ and θ in the horizontal plane. Similarly for δ_T, δ_L and F_2. Then

$$Mv\dot{\theta} = T(\delta-\delta_T) - Mg\cos\theta_a + K_L\rho d^2v^2(\delta-\delta_L) + K_S\rho d^3v\dot{\phi} - F_2 \tag{11.10}$$

By doing the same thing for (11.7) and its companion equation for the horizontal plane, we have

$$Mk^2\ddot{\phi} = TL - K_M\rho d^3v^2(\delta-\delta_M) - K_H\rho d^4v\dot{\phi} - \dot{m}\dot{\phi}(r_e^2-k^2) - J \tag{11.11}$$

where $\ddot{\phi}$, L, δ, δ_M, $\dot{\phi}$ and J are complex numbers and all other quantities are real.

If the trajectory length s is used as the independent variable, these equations become

$$Mv\frac{dv}{ds} = T - Mg\sin\theta_a - K_D\rho d^2v^2 - F_1 \tag{11.12}$$

$$Mv^2\frac{d\theta}{ds} = T(\delta-\delta_T) - Mg\cos\theta_a + K_L\rho d^2v^2(\delta-\delta_L) + K_S\rho d^3v^2\frac{d\phi}{ds} - F_2 \tag{11.13}$$

$$Mk^2v\left[v\frac{d^2\phi}{ds^2} + \frac{dv}{ds}\frac{d\phi}{ds}\right] = TL - K_M\rho d^3v^2(\delta-\delta_M) - K_H\rho d^4v^2\frac{d\phi}{ds}$$
$$- \dot{m}v\frac{d\phi}{ds}(r_e^2-k^2) - J \tag{11.14}$$

The system (11.12), (11.13) and (11.14) will be the starting point of our stability calculation.

Motion of Artillery Rockets During Burning

Artillery rockets are characterized by very short burning period with very high thrust and moderate end velocity of about 1000 ft./sec. Then during the burning period, the thrust is the predominating force. The appropriate equation for the zeroth order calculation of the velocity is thus (11.8). Furthermore, due to the low end velocity, the propellant weight is a relatively small fraction of the initial weight of the rocket. For instance, if the effective exhaust velocity is 6,000 ft./sec., then the propellant loading ratio is only 15%. Then it is allowable to consider the weight of the rocket to be roughly constant during the burning period. That is, we take M and T to be constant. Let

$$\frac{T}{M} = G \tag{11.15}$$

Then (11.8) integrates to

$$v^2 = 2GS \tag{11.16}$$

where the integration constant is chosen to make the relation simple. If the rocket is firing on a stationary launcher, the value of S represents the true trajectory length from the launching point. But if the rocket is fired from a moving platform such as aircraft, then S is not the trajectory length from the launcher due to the initial velocity of the rocket. This fact must be borne in mind.

During burning after leaving the launcher, F_2 and J in (11.13) and (11.14) are zero. But this is not the only simplification possible with these equations. First of all, since thrust is very much larger in comparison with drag and lift, it will serve to neglect the third and the fourth term to the right of (11.13). Then by dividing the equation by T, we have

$$2s\frac{d\theta}{ds} = \delta - \delta_T - \frac{g}{G}\cos\theta_a \qquad (11.17)$$

It is seen that g/G is the reciprocal of the acceleration in g.

If we divide (11.14) by Tk^2,

$$2s\frac{d^2\phi}{ds^2} + \frac{d\phi}{ds} = \frac{L}{k^2} - 2\left(\frac{K_M\rho d^3}{k^2 M}\right)s(\delta - \delta_M) - 2d\left(\frac{K_H\rho d^3}{k^2 M}\right)s\frac{d\phi}{ds}$$
$$- \frac{4}{c}\left[\left(\frac{r_e}{k}\right)^2 - 1\right]\frac{d\phi}{ds}$$

Since the velocity of flight is not very large, we can consider the aerodynamic coefficient to be roughly constant. The air density also taken to be constant. Then we can introduce a characteristic length σ, such that

$$\sigma^2 = 4\pi^2\frac{k^2 M}{K_M\rho d^3} \qquad (11.18)$$

Now if $M \sim 1$ slug, $k \sim 1$ ft., $K_M \sim 30$, $d \sim 1/3$ ft., then σ is approximately 300 ft. If this is the characteristic length of the problem, then clearly the third and fourth term to the right of the equation for ϕ is negligible. Thus the stabilizing action is mainly contributed by the restoring moment of the lift of the tail assembling due to angle of attack. The aerodynamic and jet damping have only secondary importance. Thus

$$2s\frac{d^2\phi}{ds^2} + \frac{d\phi}{ds} = \frac{L}{k^2} - \frac{8\pi^2}{\sigma^2}s(\delta - \delta_M) \qquad (11.19)$$

From these two equations, we can now deduce a single equation in δ. To do so, divide (11.17) by $2\sqrt{s}$ and differentiate the resultant equation with respect to s. Then

$$\sqrt{s}\frac{d^2\theta}{ds^2}+\frac{1}{2\sqrt{s}}\frac{d\theta}{ds}=\frac{1}{2\sqrt{s}}\frac{d\delta}{ds}-\frac{1}{4s\sqrt{s}}\left(\delta-\delta_T-\frac{g\cos\theta_a}{G}\right)$$

Now we divide (11.19) by $2\sqrt{s}$ and substract from it the above equation, then since $\delta=\varphi-\theta$, we can unite the result as

$$\sqrt{s}\frac{d^2\delta}{ds^2}+\frac{1}{\sqrt{s}}\frac{d\delta}{ds}+\left(\frac{4\pi^2\sqrt{s}}{\sigma^2}-\frac{1}{4s\sqrt{s}}\right)\delta=\frac{L}{2k^2\sqrt{s}}+\frac{4\pi^2\sqrt{s}}{\sigma^2}\delta_M-\frac{1}{4s\sqrt{s}}\left(\delta_T+\frac{g\cos\theta_a}{G}\right) \tag{11.20}$$

This equation clearly demonstrates the fact that the controlling differential equation of the stability of artillery rocket is not an equation of constant coefficient. In fact this equation can be put into the standard form of Bessel equation by the substitution

$$S=\frac{2\pi s}{\sigma} \tag{11.21}$$

Then (11.20) becomes

$$\frac{d^2\delta}{dS^2}+\frac{1}{S}\frac{d\delta}{dS}+\left\{1-\frac{(\frac{1}{2})^2}{S^2}\right\}\delta=\frac{L\sigma}{4\pi k^2S}+\delta_M-\frac{1}{4S^2}\left(\delta_T+\frac{g\cos\theta_a}{G}\right) \tag{11.22}$$

Hence the complimentary functions of δ are Bessel functions of 1/2 order. These functions are however expressible in terms of elementary functions.

Problem 11.1

What is the differential equation for δ if the independent variable is time t instead of s ?

To show this fact explicitly, multiply (11.22) by $\sqrt{S}$, then it can be written as

$$\frac{d^2}{dS^2}(\sqrt{S}\,\delta)+\sqrt{S}\,\delta=\frac{\sigma L}{4\pi k^2\sqrt{S}}+\sqrt{S}\,\delta_M-\frac{1}{4S\sqrt{S}}\left(\delta_T+\frac{g\cos\theta_a}{G}\right) \tag{11.23}$$

The complimentary functions of this equation are then trigometric functions of δ divided by $\sqrt{S}$. Or introduce the new dependent variable Z ,

$$Z = \sqrt{S}\,\delta \tag{11.24}$$

(11.23) becomes

$$\frac{d^2 Z}{dS^2} + Z = Q(S) \tag{11.25}$$

where

$$Q(S) = \frac{\sigma L}{4\pi k^2 \sqrt{S}} + \sqrt{S}\,\delta_M - \frac{1}{4S\sqrt{S}}\left(\delta_T + \frac{g\cos\theta_a}{G}\right) \tag{11.26}$$

The initial conditions of rocket after leaving the launcher is specified

as

$$V = V_p$$
$$\theta = \theta_p$$
$$\delta = \delta_p$$

and

$$\dot{\varphi} = \dot{\varphi}_p$$

Thus at

$$\left.\begin{aligned} S = P &= \frac{1}{2G} V_p^2 \frac{2\pi}{\sigma} = \frac{\pi V_p^2}{G\sigma} \\ Z &= \sqrt{P}\,\delta_p = A \end{aligned}\right\} \tag{11.27}$$

To derive the initial condition for dZ/dS, we need first to re-write (11.17)

as

$$2S\frac{d\theta}{dS} = \delta - \delta_T - \frac{g\cos\theta_a}{G}$$

Or by putting $S = P$,

$$\sqrt{P}\left(\frac{d\theta}{dS}\right)_p = \frac{\delta_p - \delta_T - \frac{g\cos\theta_a}{G}}{2\sqrt{P}}$$

Since $\theta = \phi - \delta$, we get

$$\sqrt{P}\left(\frac{d\delta}{dS}\right)_p + \frac{\delta_p}{2\sqrt{P}} = \left[\frac{d}{dS}(\sqrt{S}\,\delta)\right]_p$$
$$= \sqrt{P}\left(\frac{d\phi}{dS}\right)_p + \frac{\delta_T + g\frac{\cos\theta_a}{G}}{2\sqrt{P}}$$

Or, at $s = p$,

$$\left(\frac{dz}{ds}\right)_p = \sqrt{p}\,\frac{\sigma\dot{\phi}_p}{2\pi\sqrt{p}} + \frac{\delta_T + (g\cos\theta/G)}{2\sqrt{p}} = B \tag{11.28}$$

The solution of (11.25) with the initial conditions (11.27) and (11.28) is given by

$$\begin{aligned} Z = {} & \cos(s-p)\left[A - \int_p^s \sin(s-p)\,Q(s)\,ds\right] \\ & + \sin(s-p)\left[B + \int_p^s \cos(s-p)\,Q(s)\,ds\right] \end{aligned} \tag{11.29}$$

where $Q(s)$ is specified by (11.26). Because of the fact that $Q(s)$ contain half powers of s, Z actually involves Fresnel integrals in s. For the particular purpose on hand, Rosser and collaborators introduced special "rocket functions" to facilitate the computation. They are tabulated in Rosser's book.

Problem 11.2

Verify the solution given by (11.29).

Problem 11.3

Obtain explicit form of δ in terms of Fresnel integrals f_1 and f_2 defined as

$$f_1(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right)dt$$

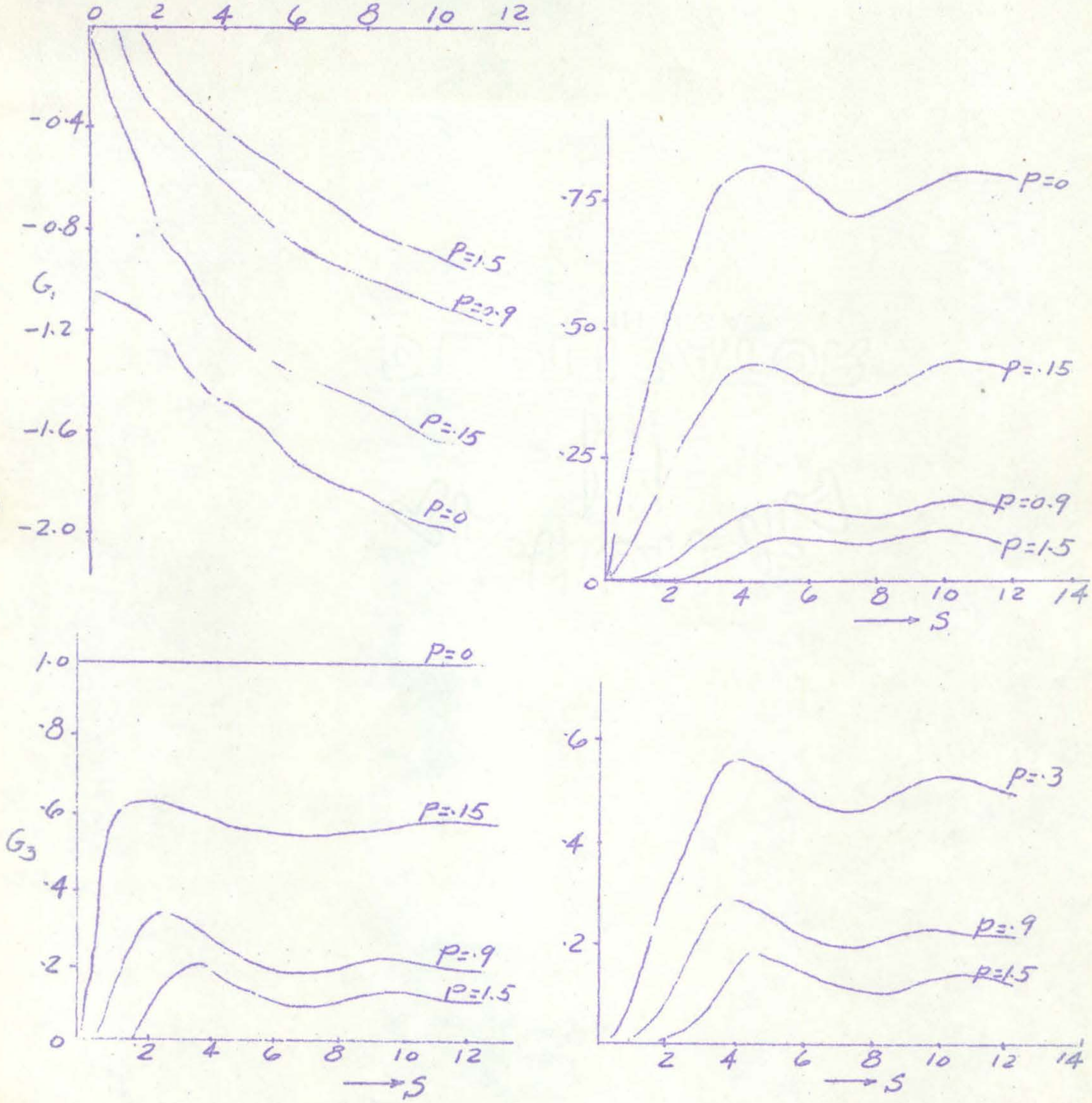
$$f_2(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right)dt$$

Trajectory Inclination Angle θ

The character of the solution can be demonstrated by considering the trajectory inclination angle θ. θ is also the most important disturbance quantity, since it can be used to estimate the dispersion. The result of calculation can be expressed as

$$\theta = \theta_p + \left(\delta_T + \frac{g\cos\theta_a}{G}\right)G_1 + \frac{\sigma L}{2\pi k^2}G_2 - \delta_M(G_1+G_3) + \delta_p G_3 + \frac{1}{2}\sqrt{\frac{\sigma}{\pi G}}\,\dot{\phi}_p\left(\frac{G_4}{\sqrt{P}}\right) \qquad (11.30)$$

where G_1, G_2, G_3 and G_4 are functions of S and P. The general charcter of these functions are sketched in the following figures:*



* Graphs of these functions are given on pp. 83-86 of Rosser, etc.

We note first that all these functions are real. Then if the real and imaginary parts of θ are separated to obtain the trajectory inclinations in the vertical plane and the horizontal plane, the quantities for the vertical plane are influended only by the initial values of θ_p . δ_p and $\dot{\phi}_p$ in the vertical plane, and the quantities for the horizontal plane are influenced only by the initial values in the horizontal plane. In other words, the motions in the vertical plane and the horizontal plane are mutually independent. This is of course to be expected as we have no first order conpling effects in our chosen equations of motion. If the rocket as spin, this will not be true.

We also note that as P increases, the magnitude of these functions decreases. Incuring P means increasing initial velocity U_p as shown by (11.27), or increased effectivness of the fin in restoring the rocket to undistur ed direction. When P is zero, there is no initial velocity, then during the first few instant, very rapid increase in the inclination can be expected. Therefore small values of P means large deviations.

The most important feature of the present result is however the fact that none of the G functions would vanish for large values of S , in spite of the asymptotically decreasing character of the Bessel functions which are the solutions of the homogeneous equation. The G_1 function is indeed unbounded for large S ($\sim \log S$). Hence if we define stability as characterized by the boundedness of all disturbances at large S (or time). Then the rocket is unstable, with stable solutions of the homogeneous solution. This situation is quite contrary to the problem of stability in systems controlled by a differential equation with constant coefficient. There the stability of the "free" oscillation or the complimentary function assures the stability of the particular integral. Hence the question of stability of systems with constant coefficient is very much simpler and can be answered broadly.

Mathematically, the difference might be considered from the following view-point: If the system is one with constant coefficient say

$$\frac{d^2y}{dx^2}+a\frac{dy}{dx}+by=f(x)$$

The "forcing" function can be expanded into a Fourier integral if it satisfies certain conditions of continuity and boundness.* Then

$$f(x)=\frac{2}{\pi}\int_0^\infty\int_{-\infty}^\infty \cos u(t-x)f(t)\,dt\,du$$

$$=\frac{1}{\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty e^{iu(t-x)}f(t)\,dt\,du$$

$$=\frac{1}{\pi}\int_{-\infty}^\infty e^{iux}du\int_{-\infty}^\infty e^{iut}f(t)\,dt$$

Therefore the solution for y is

$$y(x)=\frac{1}{\pi}\int_{-\infty}^\infty \frac{e^{-iux}du}{b-iau-u^2}\int_{-\infty}^\infty e^{iut}f(t)\,dt$$

The simplicity of the system with constant coefficient lies in the fact that the continuity and boundness of f(x)assures the continuity and boundness of y if $a>0$, $b>a$. If $a>0$, $b>a$, the complimentary functions are damped. Thus if the complimentary functions are stable, the complete solution with particular integral will be stable for all "reasonable" forcing functions.

A similar treatment for the equation of variable coefficient, say

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{(\frac{1}{2})^2}{x^2}\right)y=f(x)$$

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* See for instance, Whittaker and Watson "Modern Analysis" 9.7.

can be carried through by using the Fourier-Bessel integral* due to Henkel. Thus

$$f(x) = \int_0^\infty J_{\frac{1}{2}}(ux)u\,du \int_0^\infty f(t) J_{\frac{1}{2}}(ut)t\,dt$$

Since $J_{\frac{1}{2}}(\xi)$ satisfies the equation

$$\frac{d^2 J_{\frac{1}{2}}}{d\xi^2} + \frac{1}{\xi}\frac{dJ_{\frac{1}{2}}}{d\xi} + \left(1 - \frac{(\frac{1}{2})^2}{\xi^2}\right) J_{\frac{1}{2}} = 0$$

The solution for y can be written as

$$y(x) = \int_0^\infty \frac{J_{\frac{1}{2}}(ux)u\,du}{1-u^2} \int_0^\infty f(t) J_{\frac{1}{2}}(ut)t\,dt + A J_{\frac{1}{2}}(x) + B J_{\frac{1}{2}}(x)$$

Now, however, due to the weakly damped character of $J_{\frac{1}{2}}$ for large ξ, the boundness of $y(x)$ cannot be assured even if $f(x)$ is reasonable.

Therefore the question of stability cannot be answered by studying the complimentary functions alone. On the other hand, every forcing function will give different answers as to the stability of the motion. The problem of whether the system will perform satisfactorily has to be studied individually for each case. If we give up the attempt to determine the behavior of the system for all forcing functions, we can make a generalization in another direction: We can generalize the properties of the system and obtain a generalized approach to the problem of disturbed motion. This is the general theory of slightly perturbed motions, or the ballistic disturbance theory.

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* See N. Nielson, "Handbuch der Theorie der Cylinderfunktionen", pp. 366-370.

12. Ballistic Disturbance Theory for Guidance and Control

The object of ballistic disturbance theory is to calculate the behavior of a projectile or vehicle near the so-called normal trajectory. The normal trajectory is a certain trajectory with specified initial conditions, propulsion program, atmospheric conditions and programed lift and drag. If the actual conditions are slightly different from these specified conditions, or if the vehicle is disturbed from its normal trajectory by accidental deviations such as wind gust, the trajectory of the vehicle will be different from the normal trajectory, however, it will remain near to the normal trajectory. This fact of nearness to a calculated, known, trajectory is the basis for the linearization of the differential equations of motion for the disturbed trajectory. We will then deal with linearized equations.

The original purpose of ballistic disturbance theory is to calculate the small modification of the trajectory of a projectile due to deviations of the weight of the round from standard value, of the atmospheric conditions, and the effects of wind, etc. However with the advent of modern large and fast computing machines, the tendency was to calculate all neighboring trajectories separately. The usefulness of ballistic disturbance theory then vanishes. However the problem of control and stability of vehicles is just the problem for the ballistic disturbance theory, especially when the normal trajectory is a "ballistic" trajectory with rapidly varying aerodynamic lift and drag, weight, etc. It is the purpose of this chapter to demonstrate this statement.

The ballistic disturbance theory was developed by many authors, being for many years the field of mathematicians. The most recent book on this subject is that of R. Sänger.* But we shall adopt a formulation by

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* R. Sänger, "Ballistiche Störungstheorie" Birkhauser, Basel (1949).

F. R. Moulton* and G. A. Bliss.** In particular, the analysis follows a paper by R. Drenick.*** It is assumed that the vehicle is remotely controlled from one or more radio stations, and that the angular velocity of the vehicle is so small as to allow the neglection of angular acceleration. The moments about the center of gravity is thus always in equilibrium. The control then changes the angle of attack α through a change in trim of the vehicle.

Normal Trajectory

We shall assume that the trajectory is sufficiently short so that the earth can be considered as flat and the gravitational acceleration to be a constant g. Take x and y to be horizontal and the vertical axes, and v_x and v_y the velocities in these directions. The earth is non-rotating. The equations of motion can be written as

$$\left.\begin{aligned} \dot{x} &= v_x \\ \dot{y} &= v_y \end{aligned}\right\} \qquad (12.1)$$

and

$$\left.\begin{aligned} \dot{v}_x &= \phi_T (v_x \cos\alpha - v_y \sin\alpha) - v_x \Delta - v_y \Lambda \\ \dot{v}_y &= \phi_T (v_x \sin\alpha + v_y \cos\alpha) - v_y \Delta + v_x \Lambda - g \end{aligned}\right\} \qquad (12.2)$$

In these equations, ϕ_T , Δ , and Λ are related to thrust F_T, the drag D, and lift L of the vehicle by

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* F. R. Moulton, "New Methods in Exterior Ballistics" University of Chicago Press, Chicago, Illinois (1926).

** G. A. Bliss, "Mathematics for Exterior Ballistics" John Wiley & Sons, Inc. New York (1944).

*** R. Drenick, "The Perturbation Calculus in Missile Ballistics" Journal of the Franklin Institute, 251: 423-436 (1951).

$$\phi_r = \frac{F_r g}{W U}, \quad \Delta = \frac{D g}{W U}, \quad \Lambda = \frac{L g}{W V} \tag{12.3}$$

where W is the instantaneous weight and U is the velocity of the vehicle. Thus

$$U^2 = U_x^2 + U_y^2 \tag{12.4}$$

If A is the reference area, the drag and the lift can be computed from the drag coefficient C_D and the lift coefficient C_L by the following equations:

$$\begin{aligned} D &= \frac{\rho}{2} U^2 A C_D(\alpha, M) \\ L &= \frac{\rho}{2} U^2 A C_L(\alpha, M) \end{aligned} \tag{12.5}$$

where ρ is the air density, a function of the altitude y. The coefficients C_D and C_L are both functions of the angle of attack α and the Mach number M of flight.

The angle of attack α which enters these equations in the thrust terms as well as implicitly through the drag and the lift coefficients C_D and C_L is an undetermined parameter. Its appearance in the equations of motion constitutes the chief departure of guided missile ballistics from conventional ballistics. For, within certain limits, α can be subjected to arbitrary conditions and the flight path of the vehicle can be shaped accordingly to fulfill desired specifications.

There is one additional parameter in these equations which can be disposed of rather freely which, therefore, offers the possibility of some control over the progress of flight. It is the burning time, i.e., the time during which the thrust does not vanish. In the case of V-2 vehicle, for instance, range control was effected by a proper adjustment of the burning time. It is true that this description does not exhaust the possibilities of flight control. It does, however, cover the methods which have so far been studied thoroughly or realized in practice.

The integration of this system of non-linear differential equations is usually carried out by a computing machine. Before it is undertaken, disposition must have been made concerning the two adjustable parameters, burning time and angle of attack. The end of burning is usually effected on the basis of some requirement which will be referred to as the "cut off criterion". The angle of attack is determined as a function of time by subjecting the flight to some additional specification. This specification will be called the "guidance equation".

Once it has been settled on what basis the burning of the rocket should be terminated and in what manner the angle of attack is to be controlled during the flight, the shape of the flight path is determined. It is evident, therefore, that one should choose the cut off criterion and the guidance equation in a way which will impress upon the resulting flight path certain derived characteristics. To do this, we have to study the effects of disturbances.

Calculus of Perturbations

To apply the calculus of perturbations, we must assume that a solution of the equations has been found. This solution will be called the "standard" or "normal" trajectory. That is writing the equations of motion as

$$\left.\begin{aligned} \dot{x} &= v_x \\ \dot{y} &= v_y \\ \dot{v}_x &= F(y, v_x, v_y, \alpha, t) \\ \dot{v}_y &= G(y, v_x, v_y, \alpha, t) \end{aligned}\right\} \qquad (12.6)$$

we obtain a standard trajectory specified by

$$x = \bar{x}, \quad y = \bar{y}, \quad v_x = \bar{v}_x, \quad v_y = \bar{v}_y, \quad \alpha = \bar{\alpha} \qquad (12.7)$$

all as functions of time t. Now assume that the functions F and G contain a parameter μ - the weight of the body, say - and that the normal trajectory has been derived for $\mu = \bar{\mu}$. If the parameter now is changed somewhat to

$$\mu = \bar{\mu} + \delta\mu \tag{12.8}$$

where $\delta\mu$ is a small quentity, then a corresponding small change will occur in the path variables:

$$x = \bar{x} + \delta x, \quad y = \bar{y} + \delta y, \quad v_x = \bar{v}_x + \delta v_x, \quad v_y = \bar{v}_y + \delta v_y$$
$$\alpha = \bar{\alpha} + \delta\alpha \tag{12.9}$$

Moreover, the small perturbations will satisfy, to first order accuracy, equations

$$\left.\begin{aligned} \frac{d}{dt}(\delta x) &= \delta v_x \\ \frac{d}{dt}(\delta y) &= \delta v_y \end{aligned}\right\} \tag{12.10}$$

$$\left.\begin{aligned} \frac{d}{dt}(\delta v_x) &= \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial v_x}\delta v_x + \frac{\partial F}{\partial v_y}\delta v_y + \frac{\partial F}{\partial \alpha}\delta\alpha + \frac{\partial F}{\partial \mu}\delta\mu \\ \frac{d}{dt}(\delta v_y) &= \frac{\partial G}{\partial y}\delta y + \frac{\partial G}{\partial v_x}\delta v_x + \frac{\partial G}{\partial v_y}\delta v_y + \frac{\partial G}{\partial \alpha}\delta\alpha + \frac{\partial G}{\partial \mu}\delta\mu \end{aligned}\right\} \tag{12.11}$$

Or more briefly

$$\left.\begin{aligned} \delta\dot{x} &= \delta v_x \\ \delta\dot{y} &= \delta v_y \end{aligned}\right\} \tag{12.12}$$

$$\left.\begin{aligned} \delta\dot{v}_x &= a_1\delta y + a_2\delta v_x + a_3\delta v_y + a_4\delta\alpha + a_5\delta\mu \\ \delta\dot{v}_y &= b_1\delta y + b_2\delta v_x + b_3\delta v_y + b_4\delta\alpha + b_5\delta\mu \end{aligned}\right\} \tag{12.13}$$

The a's and b's are to be computed from the expression of F and G specified by (12.2), (12.3), (12.4) and (12.5). For instance,

$$a_1 = \frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left[\phi_T(V_x\cos\alpha - V_y\sin\alpha) - V_x\Delta - V_y\Lambda\right]$$

But

$$\frac{\partial}{\partial y}\left[\phi_T(V_x\cos\alpha - V_y\sin\alpha)\right] = (V_x\cos\alpha - V_y\sin\alpha)\frac{\partial\phi_T}{\partial y}$$

$$= (V_x\cos\alpha - V_y\sin\alpha)\frac{\partial}{\partial y}\left(\frac{F_T g}{WV}\right)$$

$$= \phi_T(V_x\cos\alpha - V_y\sin\alpha)\frac{\partial \log F_t}{\partial y}$$

The thrust can vary with respect to y at constant propellant feed rate and pressure for change in the thrust coefficient at different pressure expansion ratios of the nozzle. Furthermore

$$\frac{\partial}{\partial y}(V_x\Delta) = V_x\frac{\partial\Delta}{\partial y} = V_x\frac{\partial}{\partial y}\left(\frac{1}{2}\frac{gVA}{W}\rho c_D\right)$$

$$= V_x\frac{1}{2}\frac{gVA}{W}\left\{\frac{d\rho}{dy}c_D - \rho\frac{\partial c_D}{\partial M}\frac{V}{a^2}\frac{da}{dy}\right\}$$

$$= V_x\frac{1}{2}\frac{gVA}{W}\rho c_D\left\{\frac{1}{\rho}\frac{d\rho}{dy} - \frac{M}{c_D}\frac{\partial c_D}{\partial M}\frac{1}{a}\frac{da}{dy}\right\}$$

$$= V_x\Delta\left\{P - QN\right\}$$

where
$$P = \frac{1}{\rho}\frac{d\rho}{dy} \qquad (12.14)$$

$$Q = \frac{M}{c_D}\frac{\partial c_D}{\partial M} \qquad (12.15)$$

$$N = \frac{1}{a}\frac{da}{dy} \qquad (12.16)$$

Therefore

$$a_1 = \phi_T(V_x\cos\alpha - V_y\sin\alpha)\frac{\partial \log F_t}{\partial y} - V_x\Delta(P - QN) - V_y\Lambda(P - RN) \qquad (12.17)$$

where
$$R = \frac{M}{c_L}\frac{\partial c_L}{\partial M}$$

Now

$$a_2 = \frac{\partial}{\partial v_x}[\phi_T(v_x\cos\alpha - v_y\sin\alpha) - V_x\Delta - V_y\Lambda]$$

$$= \phi_T\cos\alpha - \phi_T(v_x\cos\alpha - v_y\sin\alpha)\frac{v_x}{v^2} - \Delta - v_x\frac{\partial\Delta}{\partial v_x} - v_y\frac{\partial\Lambda}{\partial v_x}$$

$$= \phi_T\frac{v_y}{v^2}(v_x\sin\alpha + v_y\cos\alpha) - \Delta - v_x\frac{1}{2}\frac{\rho g A}{W}\frac{\partial v C_D}{\partial v_x} - v_y\frac{1}{2}\frac{\rho g A}{W}\frac{\partial v C_L}{\partial v_x}$$

Thus $$a_2 = \phi_T\frac{v_y}{v^2}(v_x\sin\alpha + v_y\cos\alpha) - \frac{v_x^2}{v^2}\Delta(1+Q) - \frac{v_x v_y}{v^2}\Lambda(1+R) - \Delta \qquad (12.18)$$

Similarly,

$$a_3 = -\phi_T\frac{v_x}{v^2}(v_x\sin\alpha + v_y\cos\alpha) - \frac{v_x v_y}{v^2}\Delta(1+Q) - \frac{v_y^2}{v^2}\Lambda(1+R) - \Lambda \qquad (12.19)$$

and

$$a_4 = \frac{\partial F}{\partial\alpha} = -\phi_T(v_x\sin\alpha + v_y\cos\alpha) - v_x\Delta\frac{\partial\log C_D}{\partial\alpha} - v_y\Lambda\frac{\partial\log C_L}{\partial\alpha} \qquad (12.20)$$

The coefficients b_1, b_2, b_3 and b_4 can be computed in a similar way. They are

$$b_1 = \phi_T(v_x\sin\alpha + v_y\cos\alpha)\frac{\partial\log F_t}{\partial y} - v_y\Delta(P - QN) + v_x\Lambda(P - RN) \qquad (12.21)$$

$$b_2 = -\phi_T\frac{v_y}{v^2}(v_x\cos\alpha - v_y\sin\alpha) - \frac{v_x v_y}{v^2}\Delta(1+Q) + \frac{v_x^2}{v^2}\Lambda(1+R) + \Lambda \qquad (12.22)$$

$$b_3 = \phi_T\frac{v_x}{v^2}(v_x\cos\alpha - v_y\sin\alpha) - \frac{v_y^2}{v^2}\Delta(1+Q) + \frac{v_x v_y}{v^2}\Lambda(1+R) - \Delta \qquad (12.23)$$

$$b_4 = \phi_T(v_x\cos\alpha - v_y\sin\alpha) - v_y\Delta\frac{\partial\log C_D}{\partial\alpha} + v_x\Lambda\frac{\partial\log C_L}{\partial\alpha} \qquad (12.24)$$

Prob. 12.1

Verify the Equations (12.21) to (12.24) for b_1, b_2, b_3 and b_4.

The partial derivatives a_5, b_5 depend, of course, on what paramter is being identified with μ. Several possibilities which are often encountered in design studies and similar investigations are:

Variations in the drag coefficient ($\mu = C_D$):

$$a_5 = -v_x \Delta / C_D \, , \quad b_5 = -v_y \Delta / C_D \tag{12.25}$$

Variations in the lift coefficient ($\mu = C_L$):

$$a_5 = -v_y \Lambda / C_L \, , \quad b_5 = v_x \Lambda / C_L \tag{12.26}$$

Variations in the structural weight W_s, ($\mu = W_s$):

$$a_5 = -\dot{v}_x / W \, , \quad b_5 = -(\dot{v}_y + g)/W \tag{12.27}$$

Variations in the specific impulse τ ($\mu = \tau$): If W is the weight rate of flow of propellant, which is assumed to be not changed, the thrust is

$$F_t = W \tau$$

Therefore

$$a_5 = \phi_T (v_x \cos\alpha - v_y \sin\alpha)/\tau \tag{12.28}$$
$$b_5 = \phi_T (v_x \sin\alpha + v_y \cos\alpha)/\tau$$

The coefficients a_5' and b_5' are to be evaluated along the normal trajectory, using path variables of Equation (12.7). They are thus ultimately specified functions of time t. If there are more than one variable parameter μ, the differential influence of each μ should be entered separately into Eqs. (12.11) or Eqs. (12.13). When, for instance, the differential changes of α and μ are specified as functions of t, the system of Eqs. (12.12) and (12.13) then constitutes four linear equations for four unknowns δx, δy, δv_x and δv_y. In fact, if we eliminate δv_x and δv_y, we have a set of two simultaneous linear second order ordinary differential equations for δx and δy with coefficients whichvary in a complicated way with respect to time t.

Adjoint System of Differential Equations and Fundamental Formula

Let

$$\frac{dy_i}{dt} = \sum_k a_{ik} y_k + c_i \qquad (i, k = 1, \text{----} n) \tag{12.29}$$

be a system of linear differential equations for n functions $y_i(t)$ in which the coefficients a_{ik} and c_i are given functions of t. The system

$$\frac{d\lambda_i}{dt} = -\sum a_{ki} \lambda_k \qquad (i, k = 1, \text{----} n) \tag{12.30}$$

is called by Moulton and Bliss as the system adjoint to (12.29). It is formed from (12.29) by changing the sign of the elements of the matrix a_{ik}, using the columns of this matrix as rows, and ignoring the terms c_i.

Now multiply (12.29) by λ_i and (12.30) by y_i, add and then sum over i, we have

$$\frac{d}{dt} \sum_i \lambda_i y_i = \sum_i c_i \lambda_i + \sum_i \sum_k (a_{ik} \lambda_i y_k - a_{ki} \lambda_k y_i)$$

The result of the double sum is clearly zero. So

$$\frac{d}{dt} \sum_i \lambda_i y_i = \sum_i c_i \lambda_i$$

By integrating from $t = t_1$ to $t = t_2$.

$$\sum_i \lambda_i y_i \Big|_{t_2} = \sum_i \lambda_i y_i \Big|_{t_1} + \int_{t_1}^{t_2} \sum_i c_i \lambda_i \, dt \tag{12.31}$$

This is called the Fundamental Formula.

The matrix a_{ik} for our ballistic disturbance equations is

$$a_{ik} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & a_1 & a_2 & a_3 \\ 0 & b_1 & b_2 & b_3 \end{pmatrix}$$

and
$$c_1 = c_2 = 0$$
$$c_3 = a_4\delta\alpha + a_5\delta\mu$$
$$c_4 = b_4\delta\alpha + b_5\delta\mu$$

Therefore the system of adjoint equations is

$$\left.\begin{aligned} \dot{\lambda}_1 &= 0 \\ -\dot{\lambda}_2 &= \qquad\quad a_1\lambda_3 + b_1\lambda_4 \\ -\dot{\lambda}_3 &= \lambda_1 \qquad + a_2\lambda_3 + b_2\lambda_4 \\ -\dot{\lambda}_4 &= \qquad \lambda_2 + a_3\lambda_3 + b_3\lambda_4 \end{aligned}\right\} \tag{12.32}$$

The fundamental formula now gives

$$\left[\lambda_1\delta x + \lambda_2\delta y + \lambda_3\delta v_x + \lambda_4\delta v_y\right]_{t=t_2} = \left[\lambda_1\delta x + \lambda_2\delta y + \lambda_3\delta v_x + \lambda_4\delta v_y\right]_{t=t_1} + \int_{t_1}^{t_2}\left[(\lambda_3 a_4 + \lambda_4 b_4)\delta\alpha + (\lambda_3 a_5 + \lambda_4 b_5)\delta\mu\right]dt \tag{12.33}$$

where t_1 and t_2 are two arbitrary times of flight on the standard trajectory.

These relations are general relations, without specific contents. For instance, the λ's are so far only characterized by the differential equations. The actual values of λ as functions of t depend yet on what initial values we choose for them. The proper choice of the initial values for λ depend however upon the purpose of their application, as will be seen presently.

Cut off Criterion

In the case of the ground-to-ground missilies, the chief requirement is that of accuracy of hitting. That is, the flight should be so controlled that it should end at the target regardless of variations from one flight to the next. This implies, among other things, that the cut off of the rocket power should be effected at exactly the correct instant. A precise cut off timing is, of course, of particular importance for missiles of V-2 type, which are no longer controlled after cut off. But it is also quite critical in the case of vehicles in which control does not cease simultaneously with the burning. The correction of cut off errors requires, in general, a good deal of maneuvering which must take place during the free flight, that is, in a virtual vacuum, where effective steering is either impossible or highly impractical.

The timing of the power cut off would seem at first a rather straightforward matter. The range at which a target is located determines the standard trajectory along which the target is to be reached. Its usefulness is, however, somewhat hypothetical; for the probability that a vehicle under actual flight conditions would ever follow that standard trajectory is extremely small. Fluctuations in the performance of the rocket motor, for instance, will cause it to travel a path a good deal different from the expected one and, <u>at the time of standard cut off</u>, the vehicle will in general be neither at the desired point ($\bar{x}_c$, $\bar{y}_c$) nor travel with the desired velocity ($\bar{v}_{xc}$, $\bar{v}_{yc}$). Rather, its coordinates (x_c, y_c) and velocity components (v_{xc}, v_{yc}) will differ from the anticipated ones by δx_c, δy_c, δv_{xc}, δv_{yc}:

$$x_c = \bar{x}_c + \delta x_c, \quad y_c = \bar{y}_c + \delta y_c, \quad v_{xc} = \bar{v}_{xc} + \delta v_{xc}, \quad v_{yc} = \bar{v}_{yc} + \delta v_{yc} \tag{12.34}$$

It is now desired to relocate the cut off point in such a way that despite the departures δx_c, δy_c, δv_{xc}, δv_{yc}, the range of the missile remains unaffected.

Now let us denote the quantities at impact with target by capital leters. Then the impact point is characterized by

$$Y = 0$$

Therefore if δT denotes the deviation of impact time and δy denotes the deviation of y at the standard impact time, then

$$(\bar{v}_y)_{\bar{T}}\,\delta T + (\delta y)_{\bar{T}} = 0 \tag{12.35}$$

Similarly the variation of range X is

$$\delta X = (v_x)_T \delta T + (\delta x)_T \tag{12.36}$$

By eliminating δT from Eqs. (12.35) and (12.36), we have

$$\delta X = (\delta x)_T + \cot\omega (\delta y)_T \tag{12.37}$$

where ω is the impact inclination angle of the normal trajectory, or

$$\cot\omega = -\left(\frac{v_x}{v_y}\right)_T \tag{12.38}$$

Eq. (12.37) is now in the form of $\sum \lambda_i y_i$. That is if we set the end values of λ as

$$\left.\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= \cot\omega \\ \lambda_3 &= 0 \\ \lambda_4 &= 0 \end{aligned}\right\} \text{ at } t = T \tag{12.39}$$

Then

$$\delta X = \{\lambda_1 \delta x + \lambda_2 \delta y + \lambda_3 \delta v_x + \lambda_4 \delta v_y\}_{t=T} \tag{12.40}$$

The appropriate λ functions for our purpose are those obtained by integrating the Eqs. (12.32) backwards with the end conditions of Eq. (12.39). It is interesting to note that for all t

$$\lambda_1(t) = 1 \tag{12.41}$$

Now according to the fundamental formula (12.33)

$$\begin{aligned} \delta X &= \{\lambda_1 \delta x + \lambda_2 \delta y + \lambda_3 \delta v_x + \lambda_4 \delta v_y\}_T \\ &= \{\lambda_1 \delta x + \lambda_2 \delta y + \lambda_3 \delta v_x + \lambda_4 \delta v_y\}_{t_c} \\ &\quad + \int_{t_c}^{T} [(\lambda_3 a_4 + \lambda_4 b_4)\delta\alpha + (\lambda_3 a_5 + \lambda_4 a_5)\delta\mu]\, dt \end{aligned} \tag{12.42}$$

Now if we assume that all disturbances vanish after the time t_c , then the integral in (12.42) vanishes. Therefore the requirement for the zero variation in range is translated to the cut off point as

$$0 = \lambda_1(\bar{t}_c)\delta x_c + \lambda_2(\bar{t}_c)\delta y_c + \lambda_3(\bar{t}_c)\delta v_{xc} + \lambda_4(\bar{t}_c)\delta v_{yc} \tag{12.43}$$

Needless to say the λs are computed with the end conditions (12.39) on the basis of the properties of the normal trajectory.

(12.43) can be put into a more usefull form by combining it with (12.34),

$$\bar{J}_c = \lambda_1(\bar{t}_c)\bar{x}_c + \lambda_2(\bar{t}_c)\bar{y} + \lambda_3(\bar{t}_c)\bar{v}_{xc} + \lambda_4(\bar{t}_c)\bar{v}_{yc}$$

$$= \lambda_1(\bar{t}_c)x_c + \lambda_2(\bar{t}_c)y_c + \lambda_3(\bar{t}_c)v_{xc} + \lambda_4(\bar{t}_c)v_{yc}$$

The x_c, y_c, v_{xc} and v_{yc} are however fictitious quantities: They are the actual path variables evaluated at the standard cut off time. Now let the actual cut off time be $\bar{t}_c + \delta t_c$. Then up to first order terms,

$$x_c = x(\bar{t}_c) = x(t_c) - \left(\frac{d\bar{x}}{dt}\right)_{\bar{t}_c}\delta t_c$$

and similar relations. Therefore if we call

$$J_c = \lambda_1(\bar{t}_c)x(t_c) + \lambda_2(\bar{t}_c)y(t_c) + \lambda_3(\bar{t}_c)v_x(t_c) + \lambda_4(\bar{t}_c)v_y(t_c) \tag{12.44}$$

Then the proper cut off condition is

$$\bar{J}_c = J_c - (\lambda_1\bar{v}_x + \lambda_2\bar{v}_y + \lambda_3\bar{F} + \lambda_4\bar{G})_{t=t_c}\delta t_c \tag{12.45}$$

The quantities $\bar{J}_c$ and $(\lambda_1\bar{v}_x + \lambda_2\bar{v}_y + \lambda_3\bar{F} + \lambda_4\bar{G})$ at $\bar{t}_c$ can be pre-determined from the normal trajectory. Then with data obtained from tracking measurement, the value of the right side of (12.45) can be continuously computed. When the computed value coincides with $\bar{J}_c$, this is the instant for power plant cut off. Then the range error will be zero, provided no disturbance occured after cut off.

Guidance Equation

The cut off criterion itself is however not sufficient to ensure accurate range. In fact, the most serious difficulty arises frequently right after the release of the cut off signal. Theoretically, the cut off signal should entail the instantaneous shut-down of the power plant. Actually this is not likely to happen. There usually develops some after-burning in the motor which continues to accelerate the missile beyond the cut off and may ultimately lead to a considerable range error.

In addition, the free flight is affected by winds, variations in air density, dry coefficient, missile oscillations, etc. As a result of all this, the impact will in general still be quite uncertain no matter how carefully the cut off signal has been timed. If a high accuracy of hitting is desired, one is forced to resort to some guidance after cut off.

According to the fundamental formula (12.33),

$$\delta x = \left[\lambda_1 \delta x + \lambda_2 \delta y + \lambda_3 \delta v_x + \lambda_4 \delta v_y\right]_{t_c} + \int_{t_c}^{T} \left[(\lambda_3 a_4 + \lambda_4 b_4)\delta\alpha + (\lambda_3 a_5 + \lambda_4 b_5)\delta\mu\right] dt$$

But by satisfying the cut off criterion (12.45), we have made the first term to the right of the equation vanish. Thus in order to make the range error to vanish, all is required is that the integrand of the integral should vanish for all t between t_c and T. Hence

$$\delta\alpha = -\frac{\lambda_3 a_5 + \lambda_4 b_5}{\lambda_3 a_4 + \lambda_4 b_4} \delta\mu$$

The disturbance $\delta\mu$ is however not immediately measurable. What could be measured are generally δx , δy , δv_x , δv_y at each time instant t . However from Eq. (11.13), we have

$$\delta\mu = \frac{1}{a_5 + \beta b_5}\left[\frac{d}{dt}\delta v_x + \beta\frac{d}{dt}\delta v_y - (a_1 + \beta b_1)\delta y - (a_2 + \beta b_2)\delta v_x\right.$$

$$\left. - (a_3 + \beta b_3)\delta v_y - (a_4 + \beta b_4)\delta\alpha\right]$$

where β is a chosen function of time t . Thus

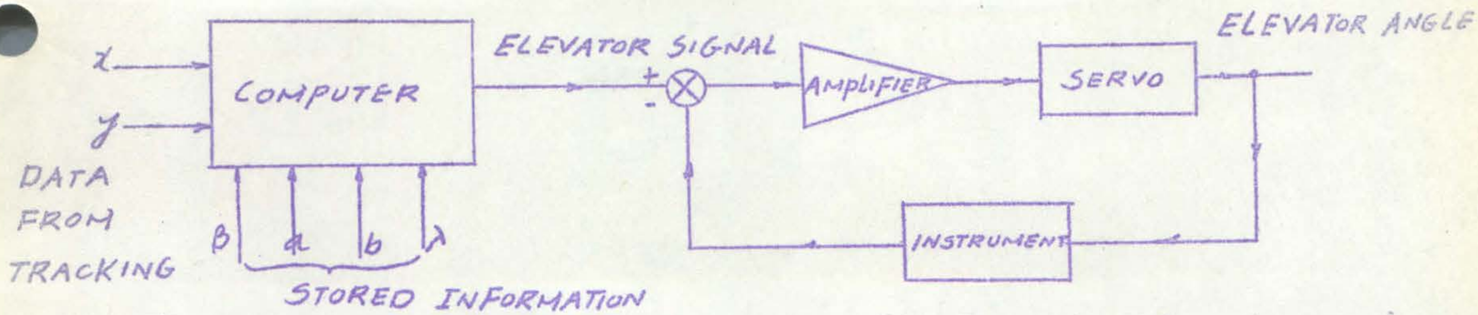
$$[(\lambda_3 a_4 + \lambda_4 b_4)(a_5+\beta b_5) - (\lambda_3 a_5+\lambda_4 b_5)(a_4+\beta b_4)]\delta\alpha$$
$$= (\lambda_3 a_5 + \lambda_4 b_5)\left[(a_1+\beta b_1)\delta y + (a_2+\beta b_2)\delta v_x + (a_3+\beta b_3)\delta v_y - \left\{\frac{d\delta v_x}{dt} + \beta\frac{d\delta v_y}{dt}\right\}\right] \tag{12.46}$$

This is the guidance equation. The choice of β must be such that with the given limitations of the accuracy of measurements on δy, δv_x, δv_y, the error in satisfying (12.46) should be the minimum possible. For instance, the choice $\beta = \lambda_4/\lambda_3$ would be certainly wrong.

The guidance equation (12.44) specifies the correction $\delta\alpha$ of the angle of attack for accurate range when the deviations of y, v_x, v_y from the normal trajectory are measured. The disturbance $\delta\mu$ does not appear in the equation. But the nature of disturbance must be known in order to fix the functions $a_5(t)$ and $b_5(t)$. Normally, the data of tracking are given as x, y, v_x and v_y. Then the deviations can be obtained from them by comparing them with the values for the computed normal trajectory. Thus

$$\delta y = y - \bar{y}$$
$$\delta v_x = v_x - \bar{v}_x$$
$$\delta v_y = v_y - \bar{v}_y$$

This information is then fed to a computer which computes $\delta\alpha$ by (12.46) with the specified a, b, λ and β. This result is then the command for changing the setting of angle of attack. This command may be translated into a command for changing the elevator angle. This command is finally taken by the elevator servo on the vehicle as an input. The output is the actual elevator angle. This part of the control is then no different from the conventional servo-mechanism.



The design criterion here are again accuracy of setting and quick response. The elevator servo is now however a purely "internal" system. The "exterior" part of the problem is taken care of by the computer. The duty of the computer is to pre-digest the tracking information and produce the correct elevator signal for zero range error, or overall "stability". The function of the computer is then quite similar to the amplifier in the simple servo, only now it is much more complicated and cannot be synthesized from pure RC-circuits.

The usefulness of the ballistic perturbation theory here is in the design of the computer - giving the guidance equation. A simple minded approach would try to make $\delta x = \delta y = 0$ at all instants. This would be however impossible to do by the available means - the adjustment of angle of attack. What is possible to do and what is sufficient to ensure accurate range, are clearly given by the ballistic disturbance theory. In this way, the present computer is different from the "simple-minded" computer based upon tracking data only, without the benefit of a, b, β and λ .

Coordinate Transformation

The particular coordinate system used in the previous analysis is particularly appropriate the problem in that the functions F and G in Eqs. (12.6) do not contain t . But this coordinate system is not the coordinate system for tracking. Therefore for practical application, we have to consider coordinate transformations.

Let

$$x = x(\xi, \eta)$$
$$y = y(\xi, \eta) \tag{12.47}$$

Then

$$\delta x = \frac{\partial \bar{x}}{\partial \xi}\delta\xi + \frac{\partial \bar{x}}{\partial \eta}\delta\eta$$
$$\delta y = \frac{\partial \bar{y}}{\partial \xi}\delta\xi + \frac{\partial \bar{y}}{\partial \eta}\delta\eta \tag{12.48}$$

Furthermore

$$\delta v_x = \left(\overline{\frac{\partial^2 x}{\partial \xi^2}}\delta\xi + \overline{\frac{\partial^2 x}{\partial \xi \partial \eta}}\delta\eta\right)\bar{\dot{\xi}} + \left(\overline{\frac{\partial^2 x}{\partial \xi \partial \eta}}\delta\xi + \overline{\frac{\partial^2 x}{\partial \eta^2}}\delta\eta\right)\bar{\dot{\eta}} + \overline{\frac{\partial x}{\partial \xi}}\delta\dot{\xi} + \overline{\frac{\partial x}{\partial \eta}}\delta\dot{\eta} \tag{12.49}$$

$$\delta v_y = \left(\overline{\frac{\partial^2 y}{\partial \xi^2}}\delta\xi + \overline{\frac{\partial^2 y}{\partial \xi \partial \eta}}\delta\eta\right)\bar{\dot{\xi}} + \left(\overline{\frac{\partial^2 y}{\partial \xi \partial \eta}}\delta\xi + \overline{\frac{\partial^2 y}{\partial \eta^2}}\delta\eta\right)\bar{\dot{\eta}} \cdot \overline{\frac{\partial y}{\partial \xi}}\delta\dot{\xi} + \overline{\frac{\partial x}{\partial \eta}}\delta\dot{\eta}$$

where

$$\dot{\xi} = \frac{d\xi}{dt}, \qquad \dot{\eta} = \frac{d\eta}{dt} \tag{12.50}$$

and the bared quantities are the quantities for the normal trajectory. With these relations, the cut off criterion and the guidance equation can be written in terms of the tracking variables, ξ, η, v_ξ and v_η.

For instance, let us consider a tracking station located on ground and in the plane of trajectory. The origin is at the tracking station, then

$$x = r\cos\phi \quad , \quad v_x = \dot{r}\cos\phi - r\sin\phi\,\dot{\phi}$$
$$y = r\sin\phi \quad , \quad v_y = \dot{r}\sin\phi + r\cos\phi\,\dot{\phi}$$

Hence

$$\delta x = \cos\bar{\phi}\cdot\delta r - \bar{r}\sin\bar{\phi}\,\delta\phi$$
$$\delta y = \sin\bar{\phi}\cdot\delta r + \bar{r}\cos\bar{\phi}\,\delta\phi$$

and

$$\delta v_x = (-\sin\bar{\phi}\cdot\delta r - \bar{r}\cos\bar{\phi}\,\delta\phi)\dot{\phi} - \bar{\dot{r}}\sin\bar{\phi}\,\delta\phi + \cos\bar{\phi}\,\delta\dot{r} - \bar{r}\sin\bar{\phi}\,\delta\dot{\phi}$$
$$\delta v_y = (\cos\bar{\phi}\,\delta r - \bar{r}\sin\bar{\phi}\,\delta\phi)\dot{\phi} + \bar{\dot{r}}\cos\bar{\phi}\,\delta\phi + \sin\bar{\phi}\,\delta\dot{r} + \bar{r}\cos\bar{\phi}\,\delta\dot{\phi}$$

Sometimes the freedom in the location of tracking station can be utilized to advantage to simplify the problem and improve the accuracy. For example the quantity J_c in (12.44) is now

$$J_c = \mu_1 r + \mu_2 \phi + \mu_3 \dot{r} + \mu_4 \dot{\phi}$$

where the μ s are again functions computed from the normal trajectory. The origin can be chosen as to make $\mu_4 = 0$. Then $\dot{\phi}$ need not be measured for the cut off criterion. This is desirable, because $\dot{\phi}$ is the most difficult one among the four to be measured accurately.

Differential Corrections for Time of Flight and Maximum Ordinate

In the previous sections, we have developed the cut off criterion and a guidance equation by considering the differential correction for range. The concept of adjoint functions λ s is, however, very flexible and is equally useful for other differential corrections.

For instance, at the terminal point, the time of flight T and the first differential δT and velocity $\bar{v}_y$ have the following relation

$$\delta T = -(\delta y)_{\bar{T}} / (v_y)_{\bar{T}} \tag{12.51}$$

Therefore for computing the differential correction for the time of flight, the appropriate end conditions for λ s are

$$\left.\begin{aligned} \lambda_1(\bar{T}) &= 0 \\ \lambda_2(\bar{T}) &= -\frac{1}{v_y(\bar{T})} \\ \lambda_3(\bar{T}) &= 0 \\ \lambda_4(\bar{T}) &= 0 \end{aligned}\right\} \tag{12.52}$$

Then equations (12.32) can be integrated using the above conditions. When the adjoint functions are computed, we have from the fundamental formula (12.33).

$$\delta T = (\lambda_1 \delta x + \lambda_2 \delta y + \lambda_3 \delta v_x + \lambda_4 \delta v_y)_{\bar{T}} = (\lambda_1 \delta x + \lambda_2 \delta y + \lambda_3 \delta v_x + \lambda_4 \delta v_y)_0$$
$$+ \int_0^{\bar{T}} [(\lambda_3 a_4 + \lambda_4 b_4)\delta\alpha + (\lambda_3 a_5 + \lambda_4 b_5)\delta\mu]\, dt \tag{12.53}$$

If the launching point is fixed and the initial velocity is zero, then the first part of the right of (12.53) vanishes. The differential correction for the time flight is given by the integral only.

As a second example, let $\bar{t}_s$ be the time of flight up to the maximum ordinate of the normal trajectory. Then the differential deviation for the maximum ordinate y_s is

$$\delta y_s = \bar{v}_y \delta t_s + \delta y \, , \qquad t = t_s$$

But at $t = t_s$, $\bar{v}_y = 0$. Therefore

$$\delta y_s = \delta y \, , \quad at \quad t = \bar{t}_s \tag{12.54}$$

Hence the appropriate "end" conditions for the adjoint functions are

$$\left.\begin{aligned} \lambda_1(\bar{t}_s) &= 0 \\ \lambda_2(\bar{t}_s) &= 1 \\ \lambda_3(\bar{t}_s) &= 0 \\ \lambda_4(\bar{t}_s) &= 0 \end{aligned}\right\} \tag{12.55}$$

And the differential deviation can be computed as

$$\begin{aligned} \delta y_s = &(\lambda_1 \delta x + \lambda_2 \delta y + \lambda_3 \delta v_x + \lambda_4 \delta v_y)_0 \\ &+ \int_0^{\bar{t}_s} [(\lambda_3 a_4 + \lambda_4 b_4)\delta\alpha + (\lambda_3 a_5 + \lambda_4 b_5)\delta\mu]\, dt \end{aligned} \tag{12.56}$$

For rockets launched at zero velocity, the first part to the right of this equation is again zero. The differential correction is then a integral only.

Prob. 12.2

What are the proper end conditions for λ functions if one wishes to calculate the differential correction for the angle of fall ω ?

Cf. G. A. Bliss, p. 87.

Effects of the Sphericity of Earth and the Rotation of Earth

In this section, we shall treat two further applications of the ballistic disturbance theory: The calculation of the effects of the sphericity of earth and the rotation of earth. Our treatment is applicable to three dimensional motion and will be thus concerned with the Earterian system x, y, z. The Cartesian trajectory is almost contained in the plane of xy. Without the effects of sphericity and rotation,

$$\left.\begin{aligned} \dot{x} &= v_x \\ \dot{y} &= v_y \\ \dot{z} &= v_z \end{aligned}\right\} \qquad \left.\begin{aligned} \dot{v}_x &= X \\ \dot{v}_y &= Y \\ \dot{v}_z &= Z \end{aligned}\right\} \tag{12.57}$$

X, Y, Z are resultant force component acting on the center of mass of the vehicle. g is the gravitational constant on the surface of earth. For non-rotating earth g is slightly different and is g_1. The disturbance equations are then

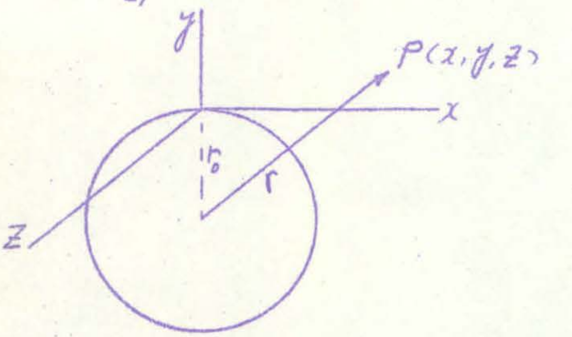
$$\left.\begin{aligned} \delta\dot{x} &= \delta v_x \\ \delta\dot{y} &= \delta v_y \\ \delta\dot{z} &= \delta v_z \\ \delta\dot{v}_x &= \frac{\partial X}{\partial x}\delta x + \frac{\partial X}{\partial y}\delta y + \frac{\partial X}{\partial z}\delta z + \frac{\partial X}{\partial v_x}\delta v_x + \frac{\partial X}{\partial v_y}\delta v_y + \frac{\partial X}{\partial v_z}\delta v_z + A \\ \delta\dot{v}_y &= \frac{\partial Y}{\partial x}\delta x + \frac{\partial Y}{\partial y}\delta y + \frac{\partial Y}{\partial z}\delta z + \frac{\partial Y}{\partial v_x}\delta v_x + \frac{\partial Y}{\partial v_y}\delta v_y + \frac{\partial Y}{\partial v_z}\delta v_z + B \\ \delta\dot{v}_z &= \frac{\partial Z}{\partial x}\delta x + \frac{\partial Z}{\partial y}\delta y + \frac{\partial Z}{\partial z}\delta z + \frac{\partial Z}{\partial v_x}\delta v_x + \frac{\partial Z}{\partial v_y}\delta v_y + \frac{\partial Z}{\partial v_z}\delta v_z + C \end{aligned}\right\} \tag{12.58}$$

where A, B and C are any disturbance sources. The partial dervitives are to be evaluated along the normal trajectory and are thus specified functions of time. The adjoint system is then

$$\left.\begin{aligned}
-\dot{\lambda}_1 &= \qquad\qquad\qquad \frac{\partial X}{\partial x}\lambda_4+\frac{\partial Y}{\partial x}\lambda_5+\frac{\partial Z}{\partial x}\lambda_6\\
-\dot{\lambda}_2 &= \qquad\qquad\qquad \frac{\partial X}{\partial y}\lambda_4+\frac{\partial Y}{\partial y}\lambda_5+\frac{\partial Z}{\partial y}\lambda_6\\
-\dot{\lambda}_3 &= \qquad\qquad\qquad \frac{\partial X}{\partial z}\lambda_4+\frac{\partial Y}{\partial z}\lambda_5+\frac{\partial Z}{\partial z}\lambda_6\\
-\dot{\lambda}_4 &= \lambda_1 \qquad\qquad +\frac{\partial X}{\partial v_x}\lambda_4+\frac{\partial Y}{\partial v_x}\lambda_5+\frac{\partial Z}{\partial v_x}\lambda_6\\
-\dot{\lambda}_5 &= \quad \lambda_2 \qquad +\frac{\partial X}{\partial v_y}\lambda_4+\frac{\partial Y}{\partial v_y}\lambda_5+\frac{\partial Z}{\partial v_y}\lambda_6\\
-\dot{\lambda}_6 &= \qquad \lambda_3 \quad +\frac{\partial X}{\partial v_z}\lambda_4+\frac{\partial Y}{\partial v_z}\lambda_5+\frac{\partial Z}{\partial v_z}\lambda_6
\end{aligned}\right\} \qquad (12.59)$$

The fundamental formula is

$$\begin{aligned}
&[\lambda_1\delta x+\lambda_2\delta y+\lambda_3\delta z+\lambda_4\delta v_x+\lambda_5\delta v_y+\lambda_6\delta v_z]_{t_2}\\
&=[\lambda_1\delta x+\lambda_2\delta y+\lambda_3\delta z+\lambda_4\delta v_x+\lambda_5\delta v_y+\lambda_6\delta v_z]_{t_1} \qquad (12.60)\\
&\quad+\int_{t_1}^{t_2}[\lambda_4 A+\lambda_5 B+\lambda_6 C]\,dt
\end{aligned}$$



Consider the effects of sphericity first. The gravitation attraction is clearly

$$g_1\cdot\frac{r_0^2}{r^2}$$

The three components in x, y, z are thus

$$-g_1\frac{r_0^2}{r^2}\frac{x}{r}, \quad -g_1\frac{r_0^2}{r^2}\frac{r_0+y}{r}, \quad -g_1\frac{r_0^2}{r^2}\frac{z}{r}$$

Since $\qquad r^2=x^2+(r_0+y)^2+z^2$

$$\left(\frac{r_0}{r}\right)^2=\left[\left(\frac{x}{r_0}\right)^2+\left(1+\frac{y}{r_0}\right)^2+\left(\frac{z}{r_0}\right)^2\right]^{-1}=1-2\frac{y}{r_0}+\cdots\cdot$$

$$\frac{x}{r}=\frac{x}{r_0}+----, \qquad \frac{r_0+y}{r}=1+0\,\frac{y}{r_0}+----$$

Therefore the three components are

$$-g_1\frac{x}{r_0}+---;\quad -g_1+2g_1\frac{y}{r_0}----;\quad -g_1\frac{z}{r_0}+----$$

If we consider z itself a disturbance quantity, then the appropriate contributions of the sphericity to the disturbance source of Eq. (12.58) are

$$\left.\begin{aligned} A &= -g_1 \frac{x}{r_0} \\ B &= +2g_1 \frac{y}{r_0} \\ C &= 0 \end{aligned}\right\} \qquad (12.61)$$

If the earth is rotating, we replace g_1 by g .

The effects of the earth rotation on the equations of motion can be obtained by a series of coordinate transformation.* If θ is astronomical latitude of the launching point, ψ the azimuth of the initial tangent plane of the trajectory, measured clockwise from the south, and Ω the angular velocity of each rotation, then additional terms to X, Y and Z are respectively the following:

X: $$\sin\psi[2\Omega v_y\cos\theta - 2\Omega v_z\sin\theta\sin\psi + \Omega^2(x\sin\psi + z\cos\psi)$$
$$+\sin\theta\cos\psi[-2\Omega v_z\cos\psi + \Omega^2((y+r_0)\cos\theta + x\sin\theta\cos\psi - z\sin\theta\sin\psi)]$$

Y: $$\cos\theta[-2\Omega(v_x\sin\psi + v_z\cos\psi) + \Omega^2((y+r_0)\cos\theta + x\sin\theta\cos\psi - z\sin\theta\sin\psi)]$$

Z: $$\cos\psi[2\Omega v_y\cos\theta + \Omega^2(x\sin\psi + z\cos\psi) + 2\Omega v_x\sin\theta$$
$$-\Omega^2\sin\theta\sin\psi((y+r_0)\cos\theta + x\sin\theta\cos\psi - z\sin\theta\sin\psi)] \qquad (12.62)$$

Since Ω is very small, being 7.2921×10^{-5} radiaus per second, and z , v_z small, we can neglect Ω^2 , , . Hence the first order effects of the sphericity and the rotation of earth give

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*Cf. Bliss, loc. cit. p.92-95.

$$A = -gx/r_0 + 2\Omega v_y \cos\theta \sin\psi$$

$$B = 2gy/r_0 - 2\Omega v_x \cos\theta \sin\psi \qquad (12.63)$$

$$C = \qquad + 2\Omega v_x \sin\theta + 2\Omega v_y \cos\theta \cos\psi$$

With these correction terms, the effects of sphericity and rotation can be calculated, once the trajectory on a flat and non-rotating earth is known.

13. Control Design by Specified Criteria

In the previous chapters, we have considered the problem of control for systems of increasing generality. But we have so far limited ourselves to the linear problem. A very general approach to the problem of control is to consider it as a system which has a free variable at the disposal of the designer. The problem of the designer is to choose a relation between this free variable and the other variables such that the other variables of the system satisfy a certain control criteria specified by the problem. For instance, the problem of servo-stabilization of the combustion in the rocket motor can be considered as a dynamic system between the propellant injection rate, the chamber pressure and the capacity of the control capacitance in the feed line. There are two differential equations relating the three variables. The problem is to choose another relation between the three variables so that the overall system satisfies the condition of stability.

In this chapter, we shall treat the problem of control and stability from this general point of view, applicable to either linear or non-linear systems. Parts of the discussion follow the paper by A. S. Boksenbom and R. Hood.*

Control Problem

An important aspect of the control-synthesis problem is a clear definition of the criterions of desired controll behavior. If a variable y is to be controlled, a reasonable criterion is that the time integral

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* A. S. Boksenbom and R. Hood, "Automatic Control Systems Satisfying certain General Criterions on Transient Behavior" NACA TN 2378 (1951).

of some function of y is to be minimum or a constant; that is

$$\int_0^{t_1} f(y)\, dt = \text{constant or minimum} \tag{13.1}$$

Or

$$\int_0^{t_1} (y-y_s)^2 dt = \text{constant or minimum} \tag{13.2}$$

where t_1 is the time at the end of transient and y_s is the setting or the desired value of y. Eq. (13.2) weights the error in y as the square and according to the time duration of that error. Another type of criterion may be that which requires a certain time duration to be a minimum or a constant; that is

$$\int_0^{t_1} dt = \text{constant or minimum} \tag{13.3}$$

The use of a single criterion, such as equation (13.1) will usually yield $f(y)$ = constant. This result is reasonable because $f(y)$ can usually be made identically a constant if no additional criteria is imposed on other variables in the system. Usually, certain limiting conditions exist, however, on other variables in the system and these conditions must be included in the original criterions.

Thus, a possible criterion could be written as follows:

$$\left.\begin{aligned} &\int_0^{t_1} (y-y_s)^2 dt = \text{minimum} \\ \text{for} \quad &\int_0^{t_1} f(z)\, dt = \text{constant} \end{aligned}\right\} \tag{13.4}$$

If, for instance, y = engine speed and z = characteristic temperature of a gas turbine engine, the criterion of (13.4) states that it is desirable to design a control system such that, for a particular value of a temperature integral, the integral of the speed-error squared is a minimum. This criterion may be used if, for instance, it is known that an over-temperature condition can be tolerated for a certain period of time, say the total heat input to the blades, and it is desired to keep the average speed error at a minimum during a transient.

The general theory will show that as many criterions as desired of the type shown in (13.1) to (13.4) can be included together and a control system can be derived that automatically satisfies all these criteria simultaneously.

Another aspect of the control criteria is the end conditions of the integrals of (13.1) to (13.4). The time interval for which these integrals are to be a minimum or a constant must be chosen. A reasonable time interval is any duration during which essential external disturbances are constant and during which the system to be controlled moves from one essential level of operation to another. The essential external disturbances are those that cannot be immediately corrected by the control system. If an essential external disturbance were allowed within the time interval of the criteria, no physically realizable system could be expected to anticipate this disturbance so as to behave properly before disturbance occur. An essential level of operation is any specific condition of only those variables that must be continuous. In the case of a turbojet engine the transient behavior of which can be described by a first order differential equation, the engine speed determines the level of operation. If the fuel system must be considered or if the temperature does not respond to speed immediately, then both the engine and the acceleration are required to describe the essential level of the engine. We shall see this presently.

Analytic Problem

The control system resulting from any design method must be physically realizable. There are two aspects to this problem. First, it is possible to set down criteria that are not realizable with any system or are incompatible with each other. If such criteria are used, the unrealizability will appear either as a requirement on the control to look ahead into the future or as an inability to satisfy the boundary conditions of some differential equation. In most cases, a clear understanding of the criteria used and of the system to be controlled will indicate incompatibilities of this sort.

The second aspect of physical realizability is purely mathematical. It is desired to derive a description (a differential equation) of the control or the controlled system that satisfies the criteria of control and all the necessary boundary conditions that arise in the derivation of this equation. Although the mathematical solution of the problem may be any derivative or integral of this differential equation, the physical solution of the problem requires the differential equation that itself satisfies the boundary conditions and for which no undetermined constants of integration exist. Thus, such forms as

$$\dot{y} = c\dot{x}$$

and

$$y = cx$$

are not necessarily interchangeable as descriptions of some part of a controlled system because the forms differ by an undetermined constant of integration. For stable linear systems, the effect of this constant becomes vanishingly small; for the general non-linear systems presented here, however, this constant must be considered.

Stability Problem

The requirement of stability is a special criterion that does not enter into the main body of the methods of this chapter. It may enter in the final steps of the method where the final differential equation describing the control system may be the integral of a higher-order differential equation that satisfies the necessary boundary condition for stability. In addition, it is always necessary to add to the controlled system a stability device that does not affect the behavior of the system as far as satisfying the other criteria is concerned. This device can be described as follows: For first order system, when

$$\left.\begin{array}{l} y = y_s \\ \dot{y} = 0 \end{array}\right\} \tag{13.5}$$

For second order systems, when

$$\left.\begin{array}{l} y = y_s\,, \quad \dot{y} = 0 \\ \qquad\qquad \ddot{y} = 0 \end{array}\right\} \tag{13.6}$$

General Theory

With the type of control criteria given previously by (12.1) to (12.4), we can formulate the control equation in the following manner: If, for such a list of criterions, one of the integrals is to be a minimum under the condition that the other integrals are to be constant, it is sufficient, according to Variational Calculus, to make

$$\int_0^{t_1} f(y)\,dt + \lambda_1 \int_0^{t_1} (y-y_s)^2\,dt + \lambda_2 \int_0^{t_1} f_0(z)\,dt + \lambda_3 \int_0^{t_1} dt = \text{Minimum}$$

Or

$$\int_0^{t_1} \left[f(y) + \lambda_1 (y-y_s)^2 + \lambda_2 f_0(z) + \lambda_3 \right] dt = \text{Minimum} \tag{13.7}$$

The λ_s' are arbitrary constants that enter into the control system as the adjustable parameters and are precisely determined by the choice of values that the constant integrals are to have.

The technique of the λ multipliers is widely used for problems of this type where one condition is to be a minimum under other restrictive conditions. Indeed, the conditions need not be in integral form and any functional or differential relation among variables can be handled in a similar manner.*

Equation (13.7) can be made very general when all possible restrictive conditions are included. In the final equations, if any one criterion is not to be used, then the corresponding $\lambda \to 0$. If any of the criteria are to be zero, then the corresponding $\lambda \to \infty$.

If the system to be controlled is of first order and of one degree of freedom (has one independent variable), then the variables y and z are such that $z = z(y, \dot{y})$. Equation (13.7) can then be written, in general, as

$$\int_0^{t_1} F(y, \dot{y})\,dt = \text{minimum} \tag{13.8}$$

where F is a continuous function of y and $\dot{y}$, and y is a continuous function of time t.

Let us consider $y(t)$ to be a solution. Then we construct a neighboring solution as $y(t) + \varepsilon\,\delta y(t)$. $\delta y(t)$ is an arbitrary function, and ε is a small parameter. The condition of (13.8) can then be stated as

$$\left[\frac{d}{d\varepsilon}\int_0^{t_1+\varepsilon\delta t_1} F(y+\varepsilon\delta y,\ \dot{y}+\varepsilon\delta\dot{y})\,dt\right]_{\varepsilon=0} = 0$$

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* See for instance, C. Lanczos, "The Variational Principles of Mechanics" Univ. Toronto Press (1946).

Or

$$\int_0^{t_1} \frac{\partial F}{\partial y} \delta y \, dt + \int_0^{t_1} \frac{\partial F}{\partial \dot{y}} \delta \dot{y} \, dt + F(t_1) \delta t_1 = 0 \qquad (13.9)$$

The variation δt_1, is due to the fact that the end point of the integral (13.8) is not fixed, but to be on the curve $y = f(t)$. This is to allow the proper boundary conditions of moving from one essential level of operation to another, as previously discussed. By partial integration, (13.9) becomes

$$\int_0^{t_1} \left[\frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right)\right] \delta y \, dt + \left(\frac{\partial F}{\partial \dot{y}}\right)_{t_1} \delta y(t_1) - \frac{\partial F}{\partial \dot{y}} \delta y(0) + F(t_1) \delta t_1 = 0$$

The relation between $\delta y(t_1)$ and δt_1 can be easily computed from the end condition.

$$\dot{y} \delta t_1 + \delta y(t_1) = f'(t_1) \delta t_1$$

Then by eliminating $\delta y(t_1)$, and then since δt is arbitrary, we have

$$\int_0^{t_1} \left[\frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right)\right] \delta y \, dt = 0 \qquad (13.10)$$

and

$$\delta t_1 \left[F(t_1) + \left(\frac{\partial F}{\partial \dot{y}}\right)_{t_1} \left\{f'(t_1) - \dot{y}(t_1)\right\}\right] - \left(\frac{\partial F}{\partial \dot{y}}\right)_0 \delta y(0) = 0 \qquad (13.11)$$

The time interval during which the criterion of (13.8) is to hold is considered as that during which the system moves from one essential operating level to another; in this case, from one definite value of y to another definite value of y. Thus

$$\left.\begin{aligned} \delta y(0) &= 0 \\ f'(t_1) &= 0 \end{aligned}\right\} \qquad (13.12)$$

Thus (13.10) and (13.11) becomes

$$\frac{\partial F}{\partial y} = \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right) \tag{13.13}$$

and

$$F(t_1) = \dot{y}(t_1)\left(\frac{\partial F}{\partial \dot{y}}\right)_{t_1} \tag{13.14}$$

Equation (13.13) need not hold at $t=0$ because $\delta y(0)=0$. The only condition that need hold at $t=0$ is that $\left(\frac{\partial F}{\partial \dot{y}}\right)_0$ is finite, which will be true if y is continuous at $t=0$. At the start of a new transient, $\dot{y}$, F, $\frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial \dot{y}}$ may be discontinuous, whereas at other points $(0<t\leq t_1)$ $\frac{\partial F}{\partial \dot{y}}$ will be continuous because of (13.13).

Equation (13.13) is the differential equation for the $y(t)$ that satisfies the original criterion of (13.8). The physical answer to the problem is the first integral of equation (13.13) which satisfies the boundary condition (13.14). This solution is

$$F(y,\dot{y}) = \dot{y}\frac{\partial F}{\partial \dot{y}} \tag{13.15}$$

By differentiating (13.15) with respect to t,

$$\frac{\partial F}{\partial y}\dot{y} + \ddot{y}\frac{\partial F}{\partial \dot{y}} = \ddot{y}\frac{\partial F}{\partial \dot{y}} + \dot{y}\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right)$$

Or whenever y, $\frac{\partial F}{\partial \dot{y}}$ and so forth are continuous

$$\dot{y}\left[\frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right)\right] = 0$$

Thus either $\dot{y}=0$ (which is only true during stability) or equation (13.13) is satisfied.

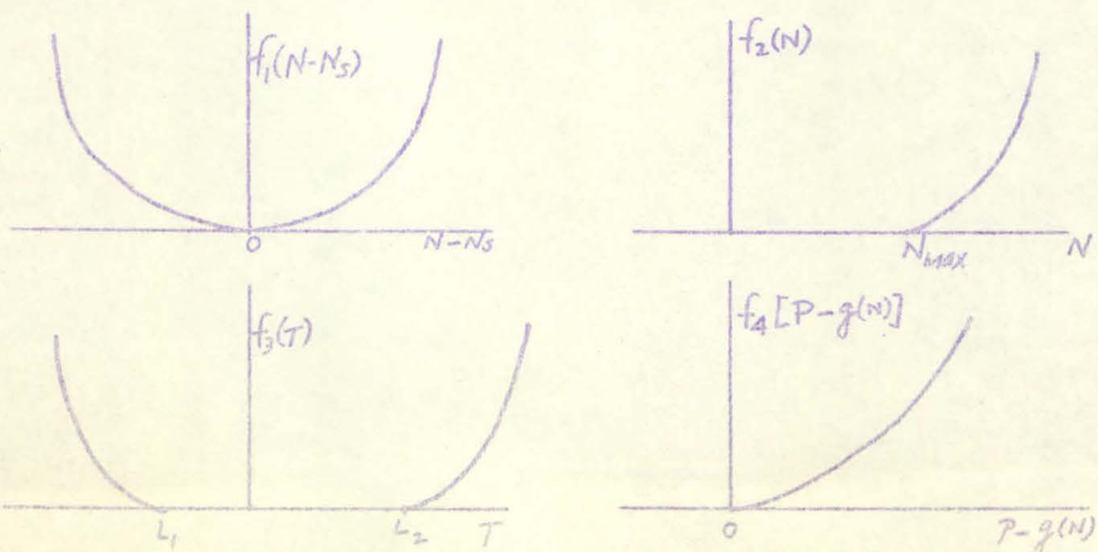
Thus Eq. (13.15) is the description of that physically realigable system the behavior of which will automatically and simultaneously satisfy those criteria included in the function F during that time interval for which the external disturbances are constant and during which the system goes from one operating level to any other operating level. A stability device must be added to the system; the description of such an ideal device is that of Eq. (13.5).

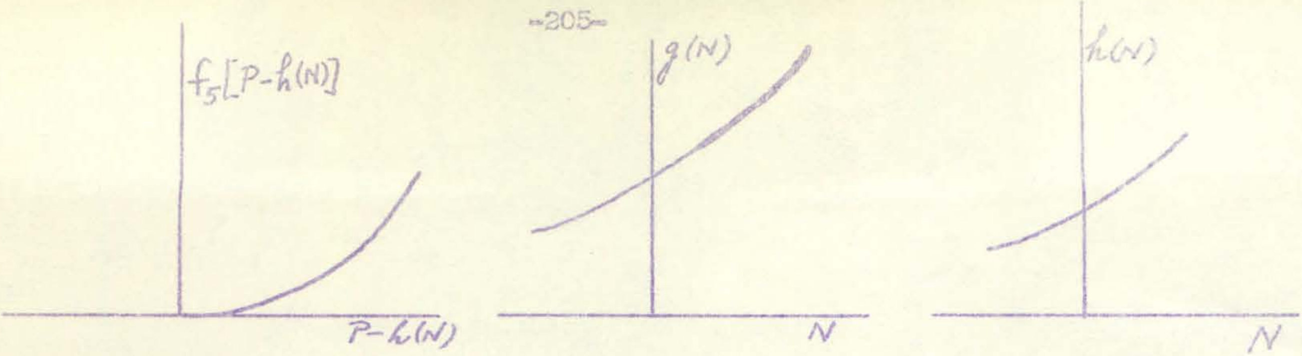
Application to Turbojet-Engine Controls

In the usual case of designing turbojet-engine controls, the engine speed that sets the essential operating level of the engine and, in the main, other pertinent characteristics such as thrust, is to be set or controlled. Limiting conditions of the engine are those of overspeed, overtemperature, compressor surge, and rich burner blow-out. The following criteria on the behavior of this engine are typical:

$$\left.\begin{array}{ll} \int_0^{t_1} f_1(N-N_s)\,dt & \text{; for speed control} \\ \int_0^{t_1} f_2(N)\,dt & \text{; for speed overshoot} \\ \int_0^{t_1} f_3(T)\,dt & \text{; for temperature overshoot and undershoot} \\ \int_0^{t_1} f_4[P-g(N)]\,dt & \text{; for compressor surge} \\ \int_0^{t_1} f_5[P-h(N)]\,dt & \text{; for blow out} \\ \text{and} & \\ \int_0^{t_1} dt & \text{; for rise time} \end{array}\right\} \quad (13.16)$$

The nature of these functions are sketched in the following figures:





P here the actual compressor discharge pressure. The quantity P-g(N) is the amount the compressor-discharge pressure exceeds the safe pressure for surge and g(N) is the compressor-discharge pressure for each engine speed at a safe value below surge. Rich burner blow out can be handled in a similar manner. The rise time is the total time for the system to move from one essential operating level to the other.

The linearized engine characteristics can be expressed, by assuming first order behavior, as follows:

$$\begin{aligned} T &= aN + a\sigma\dot{N} \\ P &= bN + c\dot{T} \end{aligned} \qquad (13.17)$$

σ is the gas turbine time constant. By substituting these relations to the integrals of Eq. (13.16), they are all of the form

$$\int_0^{t_1} f(N, \dot{N})\, dt$$

where f is a continuous function of N and $\dot{N}$, and N a continuous function of t.

<u>Speed Control with Temperature-limiting Criteria</u>

If only the errór in speed control is considered important, the criterion becomes

$$\int_0^{t_1} f_1(N - N_s)\, dt = \text{minimum}$$

Then the control condition (13.15) simply gives

$$f_1(N - N_s) = 0$$

Then because of the nature of the function f_1 , $N = N_S$. This result means that, in the absence of other criteria on the engine behavior, this speed control should keep speed error identically zero, which is physically possible only in the sense of allowing infinite temperatures. This result, however, is inconsistent with the previous development of (13.16) in that N is now a discontinuous function of time. This instance is actually a trivial case of the general problem. The result does indicate that a criterion like that must be accompanied by an additional criterion to give a physically realigable system.

Now if the error in speed control is to be combined with the condition on the overshoot and undershoot of temperature, then

$$\int_0^{t_1} [f_1(N-N_S) + \lambda f_3(T)]\, dt = \text{minimum} \qquad (13.18)$$

where $F = f_1(N-N_S) + \lambda f_3(T)$. Eq. (13.16) then becomes with (13.17)

$$f_1(N-N_S) + \lambda f_3(T) = \lambda a \sigma \dot{N} f_3'(T) \qquad (13.19)$$

The ideal stability device is such that when

$$N = N_S \qquad (13.20)$$

then $\dot{N} = 0$

Eqs.(13.19 and (13.20) describe the complete control system. Therefore we can visualize a computer so designed that it takes information from the measurements on N, T and stored information on λ , a , σ and relation between fuel rate and N, T; and thus generate the signal for the proper fuel rate in accordance with (13.19). Then shortly before N reaches N_S , the stability device takes over, so that at the end of transient, (13.20) is satisfied. In general, the control equation (13.19) is non-linear, and the computer is not a RC circuit.

It is convenient to let $f_3(T) = (T - L_2)^n$ for $T > L_2$ and $f_3(T) = (L_1 - T)^n$ for $T < L_1$. In general, the power n should be > 1, because, when $n < 1$, T may be infinite and of such nature as to make N discontinuous and physically unreal even though the integral

$$\int_0^{t_1} f_3(T)\,dt$$

is finite. In the example of this discussion, let $n = 2$ and $f_1(N - N_S) = (N - N_S)^2$. Then Eq. (13.19) becomes

$$\frac{(N - N_S)^2}{\lambda} + (L - aN)^2 = a^2\sigma^2\dot{N}^2 \qquad (13.21)$$

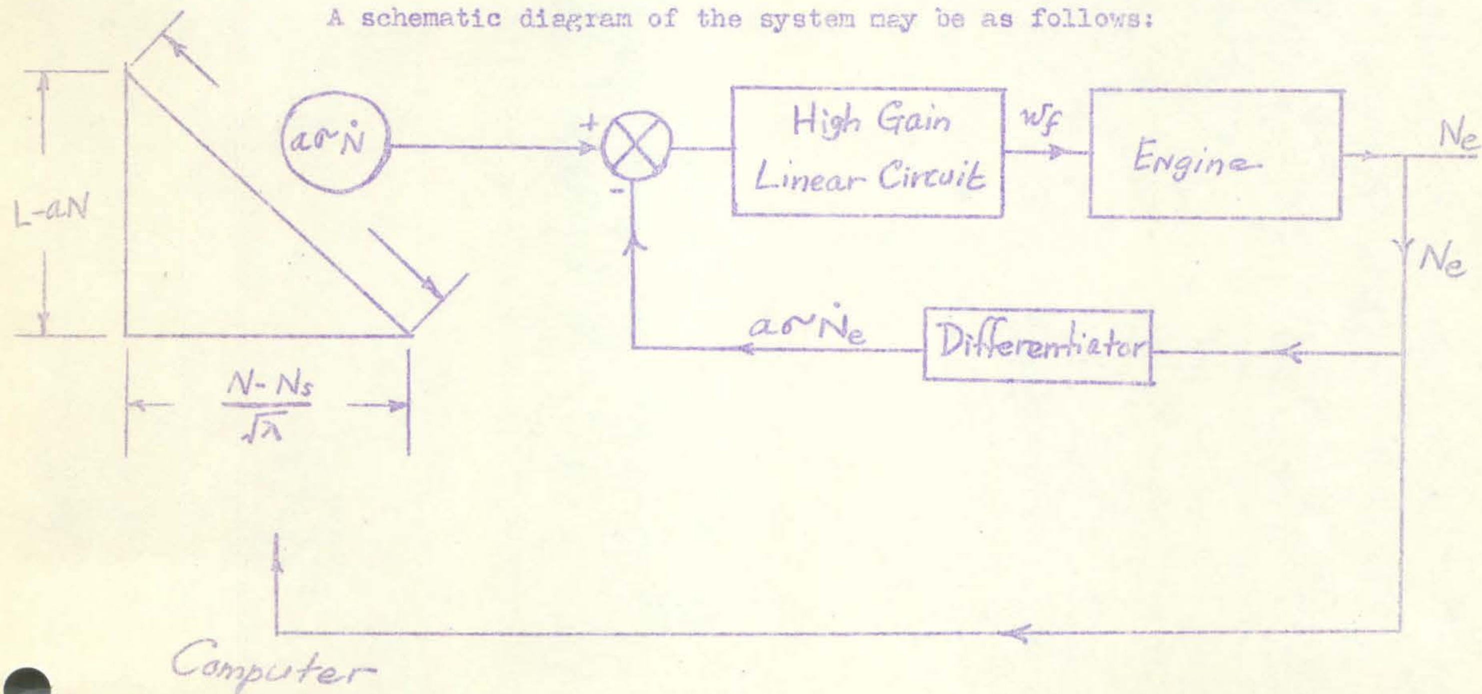
where, for acceleration, when $N < N_S$, then

$$\dot{N} > 0, \text{ and } L = L_2 \qquad (13.22)$$

For deceleration, $N > N_S$

$$\dot{N} < 0, \text{ and } L = L_1 \qquad (13.23)$$

A schematic diagram of the system may be as follows:



The actual $\dot{N}$ can be made very close to $\dot{N}$ by using a high-gain proportional controller. Provision must be made to change the sign of N and the value of L when $N - N_s$ changes sign. In addition, the stability condition requires a provision for making $\dot{N} = 0$ whenever speed error is small or zero.

The control system has one adjustable parameter λ. For any value of λ, this system will, for the value of integral temperature-overshoot obtained, give the minimum value of the integral speed-error squared. The value of λ determines the actual value of the integral temperature-overshoot. Let us consider <u>the special case</u> where $aN_s = L$; that is, acceleration or deceleration to the speed that corresponds to the limiting temperatures, according to Eq. (13.17). In this special case, (13.21) becomes

$$E\,(L - aN) = a\sigma\dot{N} \tag{13.24}$$

where

$$E = \left(1 + \frac{1}{a^2\lambda}\right)^{1/2} \tag{13.25}$$

Now the integrals can be easily computed: For instance

$$\int_0^{t_1} (T-L)^2 dt = \int_0^{t_1} \left[aN - L + a\sigma\dot{N}\right]^2 dt$$

$$= (E-1)^2 \int_0^{t_1} (L - aN)^2 dt = a^2(E-1)^2 \int_0^{t_1} (N_s - N)^2 dt$$

$$= a^2(E-1)^2 \int_{N_0}^{N_s} (N_s - N)^2 \frac{dN}{\dot{N}} = a^3\sigma(E-1)^2 \int_{N_0}^{N_s} \frac{(N_s-N)^2 dN}{Ea\,(N_s-N)}$$

$$= a^2\sigma \frac{(E-1)^2}{E} \frac{1}{2}(N_s - N_0)^2 = \frac{1}{2}\,\frac{(E-1)^2}{E}(L - aN_0)^2$$

Thus

$$\frac{1}{\sigma}\int_0^{t_1} \frac{(T-L)^2}{(L-aN_0)^2}\,dt = \frac{(E-1)^2}{2E} \tag{13.26}$$

Similarly $$\frac{a^2}{\sigma}\int_0^{t_1}\left(\frac{N-N_s}{L-aN_0}\right)^2 dt = \frac{1}{2E} \tag{13.27}$$

$$\frac{T_{max}-L}{L-aN_0} = E-1 \tag{13.28}$$

From Eq. (13.24),

$$Ea(N_s-N) = a\sigma\frac{dN}{dt}$$

So the characteristic time τ for the controlled transient is

$$\tau = \frac{\sigma}{E} \tag{13.29}$$

The left side of these equations have been put in dimensionless form. The maximum temperature T_{max} occurs at the beginning of the transient.

For $E=1$, ($\lambda=\infty$), the temperature does not overshoot, thus in agreement with our previous statement that the constant integral has the value zero when $\lambda \to \infty$. The speed integral is 0.5 and $\tau=\sigma$. As E increases, (λ decreases), the temperature integral and the maximum temperature increases, whereas the speed integral and the time constant decrease. A compromise value for E may be $\sqrt{2}$, or $a^2\lambda=1$. Once the choice of E or λ is settled, the specification for the control computer is fixed. The design can then proceed.

For the general case of (13.21), the calculation of the values of the integrals are somewhat more cumbersome, but a similar procedure applies for the design of the control system.

Second Order Systems of Several Degrees of Freedom

For the case of a second-order system with two degrees of freedom, Eq. (13.8) becomes

$$\int_0^{t_1} F(y,\dot{y},\ddot{y},z,\dot{z},\ddot{z})\,dt = \text{Minimum} \tag{13.30}$$

where y and z are independent functions of time. The condition to satisfy (13.30) is

$$\left.\frac{d}{d\varepsilon}\int_0^{t_1+\varepsilon\delta t_1} F(y+\varepsilon\delta y, \dot{y}+\varepsilon\delta\dot{y}, \ddot{y}+\varepsilon\delta\ddot{y}, z+\varepsilon\delta z, \dot{z}+\varepsilon\delta\dot{z}, \ddot{z}+\varepsilon\delta\ddot{z})dt = 0 \atop \text{at } \varepsilon = 0 \right\} \quad (13.31)$$

The time duration of the integral of Eq. (13.30) is that beginning at a definite time $(t=0)$ but not ending at any definite time but rather along some curves: $y=f_1(t)$; $\dot{y}=f_2(t)$, $z=g_1(t)$, $\dot{z}=g_2(t)$. The functions δy and δz are arbitrary and independent functions of time.

Performing the operation indicated by Eq. (13.31) gives

$$\int_0^{t_1}\left[\frac{\partial F}{\partial y}\delta y+\frac{\partial F}{\partial \dot{y}}\delta\dot{y}+\frac{\partial F}{\partial \ddot{y}}\delta\ddot{y}+\frac{\partial F}{\partial z}\delta z+\frac{\partial F}{\partial \dot{z}}\delta\dot{z}+\frac{\partial F}{\partial \ddot{z}}\delta\ddot{z}\right]dt+F(t_1)\delta t_1=0$$

After *integration by parts*,

$$\int_0^{t_1}\left[\frac{\partial F}{\partial y}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right)+\frac{d^2}{dt^2}\left(\frac{\partial F}{\partial \ddot{y}}\right)\right]\delta y\,dt+\int_0^{t_1}\left[\frac{\partial F}{\partial z}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{z}}\right)+\frac{d^2}{dt^2}\left(\frac{\partial F}{\partial \ddot{z}}\right)\right]\delta z\,dt$$
$$+\left[\frac{\partial F}{\partial \dot{y}}\delta y+\frac{\partial F}{\partial \ddot{y}}\delta\dot{y}-\frac{d}{dt}\left(\frac{\partial F}{\partial \ddot{y}}\right)\delta y\right]_0^{t_1}+F(t_1)\delta t_1$$
$$+\left[\frac{\partial F}{\partial \dot{z}}\delta z+\frac{\partial F}{\partial \ddot{z}}\delta\dot{z}-\frac{d}{dt}\left(\frac{\partial F}{\partial \ddot{z}}\right)\delta z\right]_0^{t_1}=0 \quad (13.32)$$

As before, the integrands of the integrals and the boundary-condition terms must vanish separately. From the geometry at the end condition

$$\left.\begin{aligned}\delta y(t_1) &= [f_1'(t_1)-\dot{y}(t_1)]\delta t_1\\ \delta\dot{y}(t_1) &= [f_2'(t_1)-\ddot{y}(t_1)]\delta t_1\\ \delta z(t_1) &= [g_1'(t_1)-\dot{z}(t_1)]\delta t_1\\ \delta\dot{z}(t_1) &= [g_2'(t_1)-\ddot{z}(t_1)]\delta t_1\end{aligned}\right\} \quad (13.33)$$

The three conditions from Eq. (13.32) then become

$$\left.\begin{aligned} \frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial F}{\partial \ddot{y}}\right) &= 0 \\ \frac{\partial F}{\partial z} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{z}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial F}{\partial \ddot{z}}\right) &= 0 \end{aligned}\right\} \tag{13.34}$$

and

$$\begin{aligned} &\left[F - \dot{y}\frac{\partial F}{\partial \dot{y}} + \dot{y}\frac{d}{dt}\left(\frac{\partial F}{\partial \ddot{y}}\right) - \ddot{y}\frac{\partial F}{\partial \ddot{y}} - \dot{z}\frac{\partial F}{\partial \dot{z}} + \dot{z}\frac{d}{dt}\left(\frac{\partial F}{\partial \ddot{z}}\right) - \ddot{z}\frac{\partial F}{\partial \ddot{z}}\right]_{t=t_1} \\ &+ f_1'(t_1)\left[\frac{\partial F}{\partial \dot{y}} - \frac{d}{dt}\left(\frac{\partial F}{\partial \ddot{y}}\right)\right]_{t=t_1} + f_2'(t_1)\left(\frac{\partial F}{\partial \ddot{y}}\right)_{t=t_1} + g_1'(t_1)\left[\frac{\partial F}{\partial \dot{z}} - \frac{d}{dt}\left(\frac{\partial F}{\partial \ddot{z}}\right)\right]_{t=t_1} + g_2'(t_1)\left(\frac{\partial F}{\partial \ddot{z}}\right)_{t=t_1} \\ &+ \delta y(0)\left[\frac{\partial F}{\partial \dot{y}} - \frac{d}{dt}\left(\frac{\partial F}{\partial \ddot{y}}\right)\right]_{t=0} + \delta \dot{y}(0)\left(\frac{\partial F}{\partial \ddot{y}}\right)_0 + \delta z(0)\left[\frac{\partial F}{\partial \dot{z}} - \frac{d}{dt}\left(\frac{\partial F}{\partial \ddot{z}}\right)\right]_{t=0} + \delta \dot{z}(0)\left(\frac{\partial F}{\partial \ddot{z}}\right)_0 = 0 \end{aligned} \tag{13.35}$$

Equation (13.34) is a system of two equations that satisfy the original criterion of Eq. (13.30). The physical solution to the problem is the pair of solutions of these equations that satisfy the boundary-condition equations of (13.35). If the first of equations of (13.34) is multiplied by $\dot{y}$ and the second by $\dot{z}$ and the equations are added, an exact derivative is formed the integral of which is as follows:

$$F - \dot{y}\frac{\partial F}{\partial \dot{y}} + \dot{y}\frac{d}{dt}\left(\frac{\partial F}{\partial \ddot{y}}\right) - \ddot{y}\frac{\partial F}{\partial \ddot{y}} - \dot{z}\frac{\partial F}{\partial \dot{z}} + \dot{z}\frac{d}{dt}\left(\frac{\partial F}{\partial \ddot{z}}\right) - \ddot{z}\frac{\partial F}{\partial \ddot{z}} = C \tag{13.36}$$

A study of the boundary-condition Eq. (13.35) then shows that the physically reasonable boundary conditions should be as follows:

$$\begin{aligned} &\text{If } \frac{\partial F}{\partial \dot{y}} \neq 0 \text{, or } \frac{\partial F}{\partial \ddot{y}} \neq K \text{, then } \delta y(0) = 0 \text{ and } f_1'(t_1) = 0 \\ &\text{If } \frac{\partial F}{\partial \ddot{y}} \neq 0 \text{, then } \delta \dot{y}(0) = 0 \text{ and } f_2'(t_1) = 0 \\ &\text{If } \frac{\partial F}{\partial \dot{z}} \neq 0 \text{, and } \frac{\partial F}{\partial \ddot{z}} \neq K \text{, then } \delta z(0) = 0 \text{ and } g_1'(t_1) = 0 \\ &\text{If } \frac{\partial F}{\partial \ddot{z}} \neq 0 \text{, then } \delta \dot{z}(0) = 0 \text{ and } g_2'(t_1) = 0. \end{aligned} \tag{13.37}$$

Then in Eq. (13.36), $C = 0$. The final solution to the problem of Eq. (13.30) is as follows:

$$F - \dot{y}\frac{\partial F}{\partial \dot{y}} + \dot{y}\frac{d}{dt}\left(\frac{\partial F}{\partial \ddot{y}}\right) - \ddot{y}\frac{\partial F}{\partial \ddot{y}} - \dot{z}\frac{\partial F}{\partial \dot{z}} + \dot{z}\frac{d}{dt}\left(\frac{\partial F}{\partial \ddot{z}}\right) - \ddot{z}\frac{\partial F}{\partial \ddot{z}} = 0 \qquad (13.38)$$

and either

$$\left.\begin{aligned} &\frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial F}{\partial \ddot{y}}\right) = 0 \\ \text{Or} \quad &\frac{\partial F}{\partial z} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{z}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial F}{\partial \ddot{z}}\right) = 0 \end{aligned}\right\} \qquad (13.39)$$

Equations (13.38) and (13.39) constitutes a system of two equations for two unknowns y and z. They answer the question of control design.

The meaning of the boundary-condition (13.37) is to define the original criteria for that duration during which the system moves from one essential operating level to another. Thus, if all conditions of (13.37) must hold, the system goes from one definite y, $\dot{y}$, z, and $\dot{z}$ to any other definite y, $\dot{y}$, z, and $\dot{z}$. (13.38) is a third order equation. (13.39) is a fourth order equation. Thus besides the four initial values y, $\dot{y}$, z, $\dot{z}$ we can still assign three values of $\dot{y}$, z, $\dot{z}$ at the final y. That is, when $y = y_s$, $\dot{y} = 0$, $z = z_s$ and $\dot{z} = 0$. A stability device must still be added to the system, so that

$$\ddot{y} = 0 \quad and \quad \ddot{z} = 0$$

at the final point.

Control Problem with Differential Equation as Auxiliary Condition

Let us consider y to be the essential variable whose performance has to be controlled. z is the variable which we put in the system so as to assure that y can be made to have the desired performance. The inherent dynamics of the system then gives one relation between y and z. This relation is in general a differential equation, say of second order,

$$g(y, \dot{y}, \ddot{y}; z, \dot{z}, \ddot{z}; t) = 0 \tag{13.40}$$

The performance of y during transient is specified, say, as

$$\int_0^{t_1} f(y)\, dt = \text{minimum} \tag{13.41}$$

A error integral, (13.2), would be an example of this type of specification. The problem is to derive a control equation satisfying both (13.40) and (13.41).

The mathematical problem is then a problem of calculus of variations with the differential equation (13.40) as an auxiliary condition. This can be again solved by using the method of Euler-Lagrange multiplier $\lambda(t)$.* That is,

$$\int^{t_1} F(y, \dot{y}, \ddot{y}; z, \dot{z}, \ddot{z}; t)\, dt = \text{minimum} \tag{13.42}$$

$$F = f(y) + \lambda(t)\, g(y, \dot{y}, \ddot{y}; z, \dot{z}, \ddot{z}; t) \tag{13.43}$$

The only noval feature of the problem is the introduction of time-varying multiplier $\lambda(t)$. The problem of (13.42) is exactly the same as that of (13.30). Therefore all the equations developed in the last section can be used. However, we have now three unknowns: y, z and λ. The three equations are (13.38), (13.39) and (13.40).

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* See for instance, "Vorlesungen übbr Variationsrechnung" by O. Bolza Chapter 11, (Turbner), 1909.

Concluding Remarks

The present chapter discusses the control problem from a different point of view: Instead of studying the performance of a designed system, we design the system such that it will perform according to a specified criterion. The result of analysis gives a control equation. The control equation is the design specification of the computer. The computer, necessarily of extreme speed and thus must be electro-mechanical design, takes information from actual measurements of variables at each time instant, combine it with stored information, and then generates, almost instantaneously, the proper control signal. If the system is made to follow this control signal from the computer, the performance of the system is guaranteed to be that specified. The present approach to the control problem is thus more positive than the stability analysis of the earlier chapters.

The practicability of this method of control design rests, of course, on how critical is the performance of the system and on how complex and how reliable is the electro-mechanical computer. In a great number of problems, the new control method may well be a luxury. In other problems, such as ballistic guidance problems, the new method is the only logical approach.

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Automatic Navigation of a Long Range Rocket Vehicle

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The flight of a rocket vehicle in the equatorial plane of a rotating earth is considered with possible disturbances in the atmosphere due to changes in density, in temperature, and in wind speed. These atmospheric disturbances together with possible deviations in weight and in moment of inertia of the vehicle tend to change the flight path away from the normal flight path. The paper gives the condition for the proper cut-off time for the rocket power, and the proper corrections in the elevator angle so that the vehicle will land at the chosen destination in spite of such disturbances. A scheme of tracking and automatic navigation involving a high-speed computer and elevator servo is suggested for this purpose.

THE behavior of a vehicle flying through air is closely dependent upon the aerodynamic forces acting upon the vehicle. If, during one period of oscillation of the vehicle, there is appreciable variation of the response of aerodynamic forces to the attitude of the vehicle through variations in speed, in aerodynamic coefficients, in air density, etc., then the behavior of the disturbed flight path cannot be described by a linear differential equation of constant coefficients. In fact, the basic differential equation actually has coefficients that are specified functions of time. A very simple example of such motion is that of an artillery rocket during burning of the propellant grain. As shown by J. B. Rosser, R. R. Newton, and G. L. Gross (1),[3] the basic differential equation for this particular case can be written as Bessel's differential equation for the order 1/2. The general character of the solutions of such differential equations is quite different from the character of solutions of differential equations with constant coefficients. For instance, while for equations with constant coefficients the stability of solutions for the homogeneous equation is generally sufficient to insure the stability of solutions with reasonable forcing functions, this simple state of affairs no longer prevails for equations with variable coefficients. The present theory of control and stability is built almost exclusively upon the theory of differential equations with constant coefficients. Therefore to study the disturbed motion of rockets, new methods have to be used.

R. Drenick in a recent paper (2) demonstrated the usefulness of ballistic disturbance theory in solving the control and guidance problem of ballistic trajectories described by equations with time-varying coefficients. His theory is based upon the method of adjoint functions, first introduced by G. A. Bliss (3) during World War I. The purpose of the present paper is to make Drenick's theory more complete and definite and to apply it to the problem of automatic navigation of a long-range winged rocket vehicle. A system of control involving a fast computer is suggested, whereby the range errors due to changing atmospheric conditions and deviation from the standard weight of the vehicle are automatically corrected. The main objective here is not, however, to give the final design of such an automatic navigation system, but rather to show the power of the ballistic disturbance theory for solving such problems and the various elements necessary for such a navigational system.

Equations of Motion

In order not to complicate matters, the vehicle is assumed to move in the equatorial plane of the rotating earth (Fig. 1). The planar motion is possible due to the absence of cross Coriolis force in the equatorial plane. The co-ordinate system is fixed with respect to the rotating earth, i.e., actually it rotates with the angular velocity Ω, the speed of earth rotation. The value of Ω is as follows:

$$\Omega = 7.2921 \times 10^{-5} \text{ rad/sec} \qquad [1]$$

In the equatorial plane, the position of the vehicle at any time instant t is specified by the radius r and angle θ from the starting point of the vehicle. r_0 is the mean earth radius, its value is

$$r_0 = 20.88 \times 10^6 \text{ ft} \qquad [2]$$

If g is the gravitational constant at the surface of the earth without the centrifugal force due to rotation, then

$$g = 32.2577 \text{ ft/sec}^2 \qquad [3]$$

Let R and Θ be the force per unit mass acting on the vehicle in the radial and the circumferential directions, respectively. Then the equations of motion of the center of gravity of the vehicle are

$$\left.\begin{aligned} \frac{dr}{dt} &= \dot{r} \\ \frac{d\theta}{dt} &= \dot{\theta} \\ \frac{d\dot{r}}{dt} &= R + r(\dot{\theta} \pm \Omega)^2 - g\left(\frac{r_0}{r}\right)^2 \\ r\frac{d\dot{\theta}}{dt} &= \Theta - 2\dot{r}(\dot{\theta} \pm \Omega) \end{aligned}\right\} \qquad [4]$$

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[2] Daniel and Florence Guggenheim Jet Propulsion Fellows. The computations involved in this paper were carried out by R. C. Evans, Daniel and Florence Guggenheim Jet Propulsion Fellow, and F. W. Hartwig, Captain U.S.A.F., in addition to the junior authors.

[3] Numbers in parentheses refer to the References on page 199.

where the plus sign in the second terms to the right will be valid for flights toward the east, and the minus sign for flights toward the west.

The forces acting on the center of gravity of the vehicle are the thrust f, the lift L, and the drag D (Fig. 1). Let W be the instantaneous weight of the vehicle with respect to g, and V the magnitude of air velocity relative to the vehicle. Then it is convenient to introduce the parameters Ψ, Λ, and Δ as follows:

$$\Psi = \frac{fg}{W}, \qquad \Lambda = \frac{Lg}{WV}, \qquad \Delta = \frac{Dg}{WV} \qquad [5]$$

It will be assumed that the natural wind velocity w is in the horizontal direction, positive if it is a head wind. w is considered as a function of altitude r. If v_r is the radial velocity and v_θ the circumferential velocity, i.e.,

$$\left.\begin{aligned} v_r &= \dot{r} \\ v_\theta &= r\dot{\theta} \end{aligned}\right\} \qquad [6]$$

then relative air velocity V is computed as

$$V^2 = v_r^2 + (v_\theta + w)^2 \qquad [7]$$

If β is the angle between the thrust line and the horizontal direction, then the components of forces R and Θ per unit mass are

$$\left.\begin{aligned} R &= \Psi \sin \beta + (v_\theta + w)\Lambda - v_r\Delta \\ \Theta &= \Psi \cos \beta - v_r\Lambda - (v_\theta + w)\Delta \end{aligned}\right\} \qquad [8]$$

If N is the moment of forces about the center of gravity, divided by the moment of inertia of the vehicle, the equation for the angular acceleration is

$$\frac{d\dot{\beta}}{dt} = \frac{d\dot{\theta}}{dt} + N \qquad [9]$$

To completely specify the motion of the vehicle, the lift L, the drag D, and the moment m about the center of gravity have to be given as functions of time. According to the aerodynamic convention, the L and D will be expressed in terms of the lift coefficient C_L and the drag coefficient C_D as follows:

$$\left.\begin{aligned} L &= \frac{1}{2}\rho V^2 S C_L \\ D &= \frac{1}{2}\rho V^2 S C_D \end{aligned}\right\} \qquad [10]$$

where ρ is the air density, a function of the altitude r, and S is a fixed reference area, say the wing area of the vehicle. In the present problem, since the motion of the vehicle is restricted to the equatorial plane, the attitude of the vehicle essential for aerodynamic calculations is determined by the angle of attack[4] α, i.e., the angle between the thrust line, or body axis and the relative air velocity vector (Fig. 1). The control on the motion of the vehicle is affected, however, through the elevator angle ϵ. The parameters which will affect C_L and C_D are thus α and ϵ. In addition, the aerodynamic coefficients are functions of the Reynolds number Re and the Mach number M. Thus

$$\left.\begin{aligned} C_L &= C_L(\alpha, \epsilon, M, Re) \\ C_D &= C_D(\alpha, \epsilon, M, Re) \end{aligned}\right\} \qquad [11]$$

[4] Drenick (Ref. 2) seems to take the sense of the angle of attack α opposite to the convention of aerodynamicists.

It will be assumed that the thrust line passes through the center of gravity of the vehicle; thus the thrust gives no moment. Since the angular motion of the vehicle during the powered flight is expected to be slow, the jet damping moment of the rocket is negligible. The only moment acting on the vehicle is then the aerodynamic moment m. m can be also expressed as a coefficient C_M as follows:

$$m = \frac{1}{2}\rho V^2 S l C_M \qquad [12]$$

where l is a reference length. The moment coefficient C_M is again a function of the four parameters α, ϵ, M, and Re, or

$$C_M = C_M(\alpha, \epsilon, M, Re) \qquad [13]$$

If I is the moment of inertia of the vehicle, then the magnitude of N in Equation [9] is

$$N = m/I \qquad [14]$$

With the rotations defined above, the system of equations of motion is as follows:

$$\left.\begin{aligned} \frac{dr}{dt} &= v_r \\ \frac{d\theta}{dt} &= v_\theta/r \\ \frac{d\beta}{dt} &= \dot{\beta} \\ \frac{dv_r}{dt} &= \Psi \sin \beta + (v_\theta + w)\Lambda - v_r\Delta + r\left(\frac{v_\theta}{r} \pm \Omega\right)^2 - g\left(\frac{r_0}{r}\right)^2 = F \\ \frac{dv_\theta}{dt} &= \Psi \cos \beta - v_r\Lambda - (v_\theta + w)\Delta - 2v_r\left(\frac{v_\theta}{r} + \Omega\right) + \frac{v_\theta v_r}{r} = G \\ \frac{d\dot{\beta}}{dt} &= \frac{1}{r}\{\Psi \cos \beta - v_r\Lambda - (v_\theta + w)\Delta\} - 2\frac{v_r}{r}\left(\frac{v_\theta}{r} + \Omega\right) + N = H \end{aligned}\right\} \qquad [15]$$

This system of equations is a set of first order equations for the six unknowns r, θ, β, v_r, v_θ, and $\dot{\beta}$. To solve it,

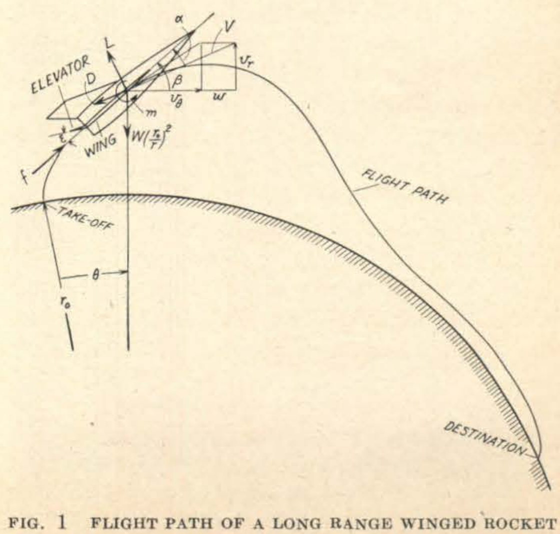

FIG. 1 FLIGHT PATH OF A LONG RANGE WINGED ROCKET
(The size of the vehicle is greatly magnified for clear graphical representation.)

the six initial values at the start, $t = 0$, for the unknowns must be specified. In addition, the thrust f, the weight W, the moment of inertia I must be given for every time instant. To determine the aerodynamic forces, the elevator angle ϵ must be specified as a function of time. The properties of the atmosphere must be known; i.e., wind velocity w, air density ρ, air viscosity and air velocity of sound must be given as functions of the altitude r. The angle of attack α of the vehicle cannot be specified; it is a quantity to be computed from the angle β and the relative air velocity vector V.

Normal Flight Path

Let the properties of the atmosphere be standardized and known as the properties of the normal atmosphere. The average characteristics of the vehicle and its power plant can be taken to be representative. Then if the elevator angle ϵ is given as a function of time, the flight path of the vehicle is determined and can be calculated by integrating the system of Equation [15]. The actual execution of this computation will be probably done on an electro-mechanical computer. This flight path of a standardized vehicle in normal atmosphere can be called the normal flight path.

The dominating feature of the normal flight path is its range. This range is the distance between the take-off point and the landing point. The problem of navigation is then to calculate the proper time for cut-off of the rocket and the proper variation of the elevator angle during flight so that the range is that desired. This problem of navigation for the standardized vehicle in normal atmosphere can be solved mathematically before the actual take-off of the vehicle, since all information for the normal flight path is known or specified beforehand.

Disturbance Equations

Natural atmospheric characteristics do not, of course, coincide with those assumed for the atmosphere. The wind velocity at each altitude changes according to the weather conditions; the temperature T is also a varying quantity. Therefore one should expect variations from the normal flight path due to changes in atmospheric conditions. The actual vehicle also may be somewhat different from the standardized vehicle, in weight, rocket performance, etc. Therefore actual flight path will be different from the normal flight path if the same elevator angle programming is used. The problem of navigation of an actual vehicle is that of correcting the elevator angle programming so that the range of the actual flight will be the same as the normal flight path and the destination is reached without error. Due to the rapidity of flight, this navigational problem cannot be solved by conventional method; but should be solved by an automatic computing system, which responds to every deviation from the normal conditions with a speed approaching instant action.

The general problem of automatic navigation is very difficult indeed. However, the deviations from the normal conditions are expected to be small, since the normal flight path is, after all, a good representation of the average situation. This fact immediately suggests that only first order quantities in deviations need be considered. This "linearization" is the basis of the ballistic disturbance theory and the present theory of automatic navigation.

Let quantities of the normal flight path be denoted by a bar over them and deviations by the δ sign. Thus for the actual flight path,

$$\left.\begin{array}{lll} r = \bar{r} + \delta r, & \theta = \bar{\theta} + \delta\theta, & \beta = \bar{\beta} + \delta\beta \\ v_r = \bar{v}_r + \delta v_r, & v_\theta = \bar{v}_\theta + \delta v_\theta, & \dot{\beta} = \bar{\dot{\beta}} + \delta\dot{\beta} \end{array}\right\} \quad \ldots [16]$$

The deviations of the actual atmosphere from the normal atmosphere are expressed as the deviation of density $\delta\rho$, the deviation of temperature δT, and the deviation of wind velocity δw; thus

$$\rho = \bar{\rho} + \delta\rho, \qquad T = \bar{T} + \delta T, \qquad w = \bar{w} + \delta w \quad [17]$$

The deviation of the actual vehicle from the normal vehicle is assumed to be limited only to deviation of weight δW and the moment of inertia δI. That is

$$W = \bar{W} + \delta W, \qquad I = \bar{I} + \delta I \ldots\ldots\ldots\ldots [18]$$

The rate of propellant flow and the effective exhaust velocity of the rocket is assumed to be standard. The wing area S and the aerodynamic characteristics of the vehicle as expressed by Equations [11] and [13] are also assumed to be invariant.

By substituting Equations [16], [17], and [18] into the equations of motion, Equation [15], and retaining only first term deviations, one has

$$\left.\begin{aligned} \frac{d\delta r}{dt} &= +\delta v_r \\ \frac{d\delta\theta}{dt} &= -\frac{\bar{v}_\theta}{\bar{r}^2}\delta r + \frac{1}{\bar{r}}\delta v_\theta \\ \frac{d\delta\beta}{dt} &= +\delta\dot{\beta} \end{aligned}\right\} \ldots\ldots [19]$$

$$\left.\begin{aligned} \frac{d\delta v_r}{dt} &= a_1\delta r + a_2\delta\beta + a_3\delta v_r + a_4\delta v_\theta + a_5\delta\epsilon + \\ &\qquad a_6\delta\rho + a_7\delta T + a_8\delta w + a_9\delta W \\ \frac{d\delta v_\theta}{dt} &= b_1\delta r + b_2\delta B\beta + b_3\delta v_r + b_4\delta v_\theta + b_5\delta\epsilon + \\ &\qquad b_6\delta\rho + b_7\delta T + b_8\delta w + b_9\delta W \\ \frac{d\delta\dot{\beta}}{dt} &= c_1\delta r + c_2\delta\beta + c_3\delta v_r + c_4\delta v_\theta + c_5\delta\epsilon + \\ &\qquad c_6\delta\rho + c_7\delta T + c_8\delta w + c_9\delta W + c_{10}\delta I \end{aligned}\right\} \ldots [20]$$

The coefficients a's, b's, and c's are partial derivatives of F, G, and H defined by Equations [15], evaluated on the normal flight path. That is, for example,

$$\left.\begin{array}{lll} a_1 = \overline{\left(\dfrac{\partial F}{\partial r}\right)}, & a_2 = \overline{\left(\dfrac{\partial F}{\partial \beta}\right)}, & a_3 = \overline{\left(\dfrac{\partial F}{\partial v_r}\right)}, \\ a_4 = \overline{\left(\dfrac{\partial F}{\partial v_\theta}\right)}, & a_5 = \overline{\left(\dfrac{\partial F}{\partial \epsilon}\right)}, & a_6 = \overline{\left(\dfrac{\partial F}{\partial \rho}\right)}, \\ a_7 = \overline{\left(\dfrac{\partial F}{\partial T}\right)}, & a_8 = \overline{\left(\dfrac{\partial F}{\partial w}\right)}, & a_9 = \overline{\left(\dfrac{\partial F}{\partial W}\right)} \end{array}\right\} \ldots [21]$$

These coefficients are calculated in detail and given in the Appendix.

The system of Equations [19] and [20] is the system of disturbance equations. They are linear. The coefficients when evaluated on the normal flight path are finally specified functions of time. If the deviations of the atmospheric properties $\delta\rho$, δT, and δw are known, and if $\delta\epsilon$, δW, and δI are specified, then this system of differential equations determines the deviations δr, $\delta\theta$, $\delta\beta$, δv_r, δv_θ, and $\delta\dot{\beta}$ from the normal trajectory. This is the direct problem in the ballistic disturbance theory. The problem of automatic navigation is however different from this. What is required is the function $\delta\epsilon$, correction to the elevator angle, such that the range error is zero. As suggested by Drenick (2), this navigational problem can best be solved by the method of adjoint functions of Bliss.

Adjoint Functions for Range Correction

The principle of the method of adjoint functions is as follows (3): Let $y_i(t)$, $i = 1, \ldots . . n$, be determined by a system of linear equations

$$\frac{dy_i}{dt} = \sum_{j=1}^{n} a_{ij} y_j + Y_i \qquad [22]$$

where a_{ij} are given coefficients which may be functions of the time t, and Y_i are specified functions of time. Now introduce a new set of functions $\lambda_i(t)$, called the adjoint functions to $y_i(t)$, which satisfy the following system of differential equations

$$\frac{d\lambda_i}{dt} = -\sum_{j=1}^{n} a_{ji}\lambda_j \qquad [23]$$

By multiplying Equation [22] by λ_i and Equation [23] by y_i, and then adding the resultant equations, it can be shown that

$$\frac{d}{dt}\sum_{i=1}^{n} \lambda_i y_i = \sum_{i=1}^{n} \lambda_i Y_i \qquad [24]$$

Equation [24] can be integrated from $t = t_1$ to $t = t_2$,

$$\sum_{i=1}^{n} \lambda_i y_i \Big|_{t=t_2} = \sum_{i=1}^{n} \lambda_i y_i \Big|_{t=t_1} + \int_{t=t_1}^{t=t_2} \left(\sum_{i=1}^{n} \lambda_i Y_i\right) dt \qquad [25]$$

Bliss named Equation [25] the "Fundamental Formula."

For the present problem, the y_i are the disturbance quantities, i.e.,

$$\left.\begin{array}{lll} y_1 = \delta r, & y_2 = \delta\theta, & y_3 = \delta\beta \\ y_4 = \delta v_r, & y_5 = \delta v_\theta, & y_6 = \delta\dot{\beta} \end{array}\right\} \qquad [26]$$

Then, according to Equation [20], the adjoint functions λ_i satisfy the following differential equations

$$\left.\begin{array}{llll} -\dfrac{d\lambda_1}{dt} = & -\dfrac{\bar{v}_\theta}{\bar{r}^2}\lambda_2 & + a_1\lambda_4 + b_1\lambda_5 + c_1\lambda_6 \\ -\dfrac{d\lambda_2}{dt} = 0 & & \\ -\dfrac{d\lambda_3}{dt} = & & a_2\lambda_4 + b_2\lambda_5 + c_2\lambda_6 \\ -\dfrac{d\lambda_4}{dt} = \lambda_1 & & + a_3\lambda_4 + b_3\lambda_5 + c_3\lambda_6 \\ -\dfrac{d\lambda_5}{dt} = & \dfrac{1}{\bar{r}}\lambda_2 & + a_4\lambda_4 + b_4\lambda_5 + c_4\lambda_6 \\ -\dfrac{d\lambda_6}{dt} = & \lambda_3 & \end{array}\right\} \qquad [27]$$

The Y_i are then

$$\left.\begin{array}{l} Y_1 = Y_2 = Y_3 = 0 \\ Y_4 = a_5\delta\epsilon + a_6\delta\rho + a_7\delta T + a_8\delta w + a_9\delta W \\ Y_5 = b_5\delta\epsilon + b_6\delta\rho + b_7\delta T + b_8\delta w + b_9\delta W \\ Y_6 = c_5\delta\epsilon + c_6\delta\rho + c_7\delta T + c_8\delta w + c_9\delta W + c_{10}\delta I \end{array}\right\} \qquad [28]$$

The Equations [27] do not determine the λ-functions completely. To do that, a set of values for λ must be specified at a certain instant. For the problem of automatic navigation, the requirement is vanishing range error; then the required boundary conditions for Equations [27] can be determined as follows: If t_2 is the time instant of landing of the actual vehicle, $\bar{t}_2$ the time instant of landing of the normal flight path, then

$$t_2 = \bar{t}_2 + \delta t_2 \qquad [29]$$

Similarly with the subscript 2 denoting the quantities at the landing instant

$$\left.\begin{array}{l} r_2 = \bar{r}_2 + \delta r_2 \\ \theta_2 = \bar{\theta}_2 + \delta\theta_2 \end{array}\right\} \qquad [30]$$

But

$$\left.\begin{array}{l} \delta r_2 = (\bar{v}_r)_{t=\bar{t}_2}\,\delta t_2 + (\delta r)_{t=\bar{t}_2} \\ \delta\theta_2 = \dfrac{1}{r_0}(\bar{v}_\theta)_{t=\bar{t}_2}\,\delta t_2 + (\delta\theta)_{t=\bar{t}_2} \end{array}\right\} \qquad [31]$$

δr_2 is, however, zero by definition, because landing means contact with the surface of earth, $\bar{r}_2 = r_2 = r_0$. By eliminating δt_2 from the Equations [31],

$$\delta\theta_2 = \left[-\frac{1}{\bar{r}}\left(\frac{\bar{v}_\theta}{\bar{v}_r}\right)\delta r + \delta\theta\right]_{t=\bar{t}_2} \qquad [32]$$

Therefore if the magnitudes of λ_i at the landing instant $t = \bar{t}_2$ are specified as

$$\left.\begin{array}{l} \lambda_1 = -\dfrac{1}{\bar{r}}\left(\dfrac{\bar{v}_\theta}{\bar{v}_r}\right) \\ \lambda_2 = 1 \\ \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0 \end{array}\right\} t = \bar{t}_2 \qquad [33]$$

then the error in range is given by

$$\delta\theta_2 = \sum_{i=1}^{n} \lambda_i y_i \Big|_{t=\bar{t}_2} = [\lambda_1\delta r + \lambda_2\delta\theta + \lambda_3\delta\beta + \lambda_4\delta v_r + \lambda_5\delta v_\theta + \lambda_6\delta\dot{\beta}]_{t=\bar{t}_2} \qquad [34]$$

When the normal flight path is determined, the coefficients in Equation [27] are specified as functions of time. These equations together with the end conditions of Equation [33] then determine the adjoint functions λ_i. The integration has to be performed "backwards" for $t < \bar{t}_2$, by perhaps an electro-mechanical computer. With the adjoint functions so determined, one can use the Fundamental Formula of Equation [25] to modify the equation for the range error given by Equation [34]: Let $\bar{t}_1$ denote the time instant for the power cut-off. Then the condition for the error $\delta\theta_2$ in range to be zero can be expressed as

$$\left.\begin{array}{l} \delta\theta_2 = 0 = [\lambda_1\delta r + \lambda_2\delta\theta + \lambda_3\delta\beta + \lambda_4\delta v_r + \lambda_5\delta v_\theta + \lambda_6\delta\dot{\beta}]_{t=\bar{t}_1} \\ \qquad + \displaystyle\int_{\bar{t}_1}^{\bar{t}_2} [\lambda_4 Y_4 + \lambda_5 Y_5 + \lambda_6 Y_6]\,dt \end{array}\right\} \qquad [35]$$

This is the basic equation for automatic navigation. It will be exploited presently.

Cut-Off Condition

The condition of Equation [35] for arbitrary disturbances can be broken down into two parts: The sum and the integral to be set to zero separately. Therefore the condition to be satisfied at the normal cut-off instant $\bar{t}_1$ is

$$[\lambda_1\delta r + \lambda_2\delta\theta + \lambda_3\delta\beta + \lambda_4\delta v_r + \lambda_5\delta v_\theta + \lambda_6\delta\dot{\beta}]_{t=\bar{t}_1} = 0 \quad [36]$$

Since the normal cut-off instant $\bar{t}_1$ is a standard time instant, but not necessarily the actual cut-off instant t_1, i.e.,

$$t_1 = \bar{t}_1 + \delta t_1 \quad [37]$$

Equation [36] should be converted into a more useful form involving the quantities at the actual cut-off instant. This is easily done, because up to the first order quantities, according to Equation [16]

$$(\delta r)_{t=\bar{t}_1} = (r)_{t=t_1} - \left(\frac{d\bar{r}}{dt}\right)_{t=\bar{t}_1}\delta t_1 - (\bar{r})_{t=\bar{t}_1}$$

Or

$$(\delta r)_{t=\bar{t}_1} = (r)_{t=t_1} - (\bar{r})_{t=\bar{t}_1} - (\bar{v}_r)_{t=\bar{t}_1}\delta t_1$$

Similarly,

$$(\delta\theta)_{t=\bar{t}_1} = (\theta)_{t=t_1} - (\bar{\theta})_{t=\bar{t}_1} - \left(\frac{1}{\bar{r}}\bar{v}_\theta\right)_{t=\bar{t}_1}\delta t_1$$

$$(\delta\beta)_{t=\bar{t}_1} = (\beta)_{t=t_1} - (\bar{\beta})_{t=\bar{t}_1} - (\dot{\beta})_{t=\bar{t}_1}\delta t_1$$

$$(\delta v_r)_{t=\bar{t}_1} = (v_r)_{t=t_1} - (\bar{v}_r)_{t=\bar{t}_1} - (\bar{F})_{t=\bar{t}_1}\delta t_1$$

$$(\delta v_\theta)_{t=\bar{t}_1} = (v_\theta)_{t=t_1} - (\bar{v}_\theta)_{t=\bar{t}_1} - (\bar{G})_{t=\bar{t}_1}\delta t_1$$

$$(\delta\dot{\beta})_{t=\bar{t}_1} = (\dot{\beta})_{t=t_1} - (\bar{\dot{\beta}})_{t=\bar{t}_1} - (\bar{H})_{t=\bar{t}_1}\delta t_1$$

where $\bar{F}$, $\bar{G}$, and $\bar{H}$, are the values of these quantities given by Equation [15] evaluated on the normal flight path. In fact, they should be evaluated at an instant just before the normal cut-off time $\bar{t}_1$ so that the accelerating force of the rocket is included and the rates of change of velocities are those of a powered flight. Now define J and $\bar{J}$ as follows,

$$J = [\lambda_1{}^*r + \lambda_2{}^*\theta + \lambda_3{}^*\beta + \lambda_4{}^*v_r + \lambda_5{}^*v_\theta + \lambda_6{}^*\dot{\beta}]_{t=t_1} \quad [38]$$

and

$$\bar{J} = [\lambda_1{}^*\bar{r} + \lambda_2{}^*\bar{\theta} + \lambda_3{}^*\bar{\beta} + \lambda_4{}^*\bar{v}_r + \lambda_5{}^*v_\theta + \lambda_6{}^*\bar{\dot{\beta}}]_{t=\bar{t}_1}$$

where $\lambda_i{}^*$ are the values of λ_i evaluated at the normal cut-off time t_1. Then the condition to be satisfied at the actual cut-off instant t_1 is

$$J = \bar{J} + \left\{\lambda_1{}^*\bar{v}_r + \lambda_2{}^*\frac{\bar{v}_\theta}{\bar{r}} + \lambda_3{}^*\bar{\dot{\beta}} + \lambda_4{}^*\bar{F} + \lambda_5{}^*\bar{G} + \lambda_6{}^*\bar{H}\right\}_{t=t_1}(t_1 - \bar{t}_1) \quad [39]$$

This is the equation to determine the proper instant of power cut-off.

When the normal flight path is known, $\bar{J}$ and the quantity within the bracket to the right of Equation [39] are fixed. Then the whole right-hand side of Equation [39] can be considered as a linearly increasing function of time t, if t is substituted for t_1. Simultaneously J can be computed at every instant before cut-off by using the predetermined $\lambda_i{}^*$ and values of position and velocity of the actual vehicle obtained by tracking stations. The magnitudes of the quantities on the two sides of Equation [39] can then be continuously compared. When they are equal to each other, condition [39] is satisfied. Then the power cut-off signal is given and the rocket power is shut off.[5]

Condition for Automatic Navigation

When the rocket power is shut off earlier or later than the normal cut-off instant $\bar{t}_1$, the propellant left in the tank, if not dumped, will alter the weight W and the moment of inertia I of the vehicle. It is also possible that the pay load of the vehicle is not that specified for the standard vehicle. Then after power-off, there is a fixed δW and δI, fixed in the sense that they do not change with time, and are known once the power cut-off is affected. Of different character are the deviations $\delta\rho$, δT, δw of the actual atmosphere from the standard atmosphere. These are not known unless they are measured. In the following, it is proposed to use the vehicle itself as a measuring instrument, and proceed as follows:

After the cut-off condition is satisfied, the condition for zero range error is that the integral in Equation [35] should vanish. Now since the Y_i in that integrand involving arbitrary disturbances $\delta\rho$, δT, and δw is not known beforehand, this condition can be satisfied only if the integrand itself vanishes. That is, according to Equations [28],

$$(\lambda_4 a_5 + \lambda_5 b_5 + \lambda_6 c_5)\delta\epsilon + (\lambda_4 a_6 + \lambda_5 b_6 + \lambda_6 c_6)\delta\rho + (\lambda_4 a_7 + \lambda_5 b_7 + \lambda_6 c_7)\delta T + (\lambda_4 a_8 + \lambda_5 b_8 + \lambda_6 c_8)\delta w + (\lambda_4 a_9 + \lambda_5 b_9 + \lambda_6 c_9)\delta W + \lambda_6 c_{10}\delta I = 0$$

Or, with the following notation

$$\left.\begin{aligned} d_5 &= \lambda_4 a_5 + \lambda_5 b_5 + \lambda_6 c_5 \\ d_6 &= \lambda_4 a_6 + \lambda_5 b_6 + \lambda_6 c_6 \\ d_7 &= \lambda_4 a_7 + \lambda_5 b_7 + \lambda_6 c_7 \\ d_8 &= \lambda_4 a_8 + \lambda_5 b_8 + \lambda_6 c_8 \\ D &= -(\lambda_4 a_9 + \lambda_5 b_9 + \lambda_6 c_9)\delta W - \lambda_6 c_{10}\delta I \end{aligned}\right\} \quad [40]$$

this condition can be written as

$$d_5\delta\epsilon + d_6\delta\rho + d_7\delta T + d_8\delta w = D \quad [41]$$

Equation [20] can be rewritten as

$$\left.\begin{aligned} a_5\delta\epsilon + a_6\delta\rho + a_7\delta T + a_8\delta w &= A \\ b_5\delta\epsilon + b_6\delta\rho + b_7\delta T + b_8\delta w &= B \\ c_5\delta\epsilon + c_6\delta\rho + c_7\delta T + c_8\delta w &= C \end{aligned}\right\} \quad [42]$$

where

$$\left.\begin{aligned} A &= \frac{d\delta v_r}{dt} - a_1\delta r - a_2\delta\beta - a_3\delta v_r - a_4\delta v_\theta - a_7\delta W \\ B &= \frac{d\delta v_\theta}{dt} - b_1\delta r - b_2\delta\beta - b_3\delta v_r - b_4\delta v_\theta - b_7\delta W \\ C &= \frac{d\delta\dot{\beta}}{dt} - c_1\delta r - c_2\delta\beta - c_3\delta v_r - c_4\delta v_\theta - c_9\delta W - c_{10}\delta I \end{aligned}\right\} \quad [43]$$

If the tracking stations for the vehicle will measure the quantities A, B, and C, then the atmospheric disturbances $\delta\rho$, δT, and δw can be determined by solving for these variations using Equation [42]. This is essentially using the vehicle itself as a measuring instrument for $\delta\rho$, δT, and δw. When $\delta\rho$, δT, and δw are known, Equation [41] gives the proper elevator angle correction $\delta\epsilon$.

A mathematically equivalent way to calculate $\delta\epsilon$

[5] Drenick's cut-off condition (Ref. 2) differs from that of Equation [39] in that the second term to the right is not present. It seems that this neglect is not justified and will introduce first order range error.

would be to directly solve for $\delta\epsilon$ using the system of Equations [41] and [42]. Thus

$$\delta\epsilon = \frac{\begin{vmatrix} A & a_6 & a_7 & a_8 \\ B & b_6 & b_7 & b_8 \\ C & c_6 & c_7 & c_8 \\ D & d_6 & d_7 & d_8 \end{vmatrix}}{\begin{vmatrix} a_5 & a_6 & a_7 & a_8 \\ b_5 & b_6 & b_7 & b_8 \\ c_5 & c_6 & c_7 & c_8 \\ d_5 & d_6 & d_7 & d_8 \end{vmatrix}} \quad \ldots\ldots[44]$$

This equation specifies the necessary change in the elevator angle at every instant to be calculated from the quantities a's, b's, c's, and A, B, C, D at the same instant. These quantities consist partly of predetermined information from the normal flight path, and partly of measured information on the position and the velocities of vehicle obtained by tracking the vehicle. At high altitudes where the air density is very small, the aerodynamic forces will be almost negligible in comparison with the gravitational and inertia forces. Then the quantities A, B, and C of Equation [43] will be the small difference of large magnitudes. These are then the quantities most difficult to determine accurately. If the actual elevator angle is made to conform with the one calculated by Equation [44], then in conjunction with the proper power cut-off as specified in the last section the vehicle will be navigated to the chosen landing point in spite of the atmospheric disturbances.

Discussion

When the general character of the flight path is chosen from the over-all engineering considerations, the first step is the calculation of the normal flight path using the properties of the standard atmosphere and the expected performance of the vehicle with normal weight. The knowledge of the normal flight path then determines the a's, b's, and c's. The Equation [27] together with the end conditions of Equation [33] allows the calculation of the adjoint functions λ_i. All this information should be on hand before the actual flight of the vehicle, and may be called the "stored data."

Before the power cut-off, the elevator angle may be programmed according to that for the normal flight path, and the stability of the vehicle is supplied by the jet vanes or by the auxiliary rockets. The tracking stations, however, go immediately into action and supply the vehicle with information on its positions and velocities. This information goes first into the cut-off computer which, using the stored information, continuously compares the magnitudes of quantities on the two sides of Equation [39], the cut-off condition. When that condition is satisfied, the power cut-off is affected.

At the instant of power cut-off, the tracking information is switched to the computer for automatic navigation. The instant of power cut-off also fixes the amount of propellant in the tank and thus determines the variations of weight δW and inertia moment δI from the standard. This information together with the stored data on the normal flight path then allows the computer to generate the elevator correction angle $\delta\epsilon$ according to Equations [40], [43], and [44]. Theoretically the value of $\delta\epsilon$ must be obtained without time delay from the instant when the information is received, because Equation [44] is a condition of equality of two quantities evaluated at identical time instants. Practically there will be time delay due to the finite computing time. However, it is now clear that this computing time must be made very short in order to satisfy the condition of automatic navigation as accurately as possible. The computed correction $\delta\epsilon$ combined with the elevator angle $\bar{\epsilon}$ determined for the normal flight path then gives the actual elevator angle setting ϵ. The design of the control mechanism for the elevator from here on follows the conventional feed-back servomechanism, with the usual criteria of quick action, stability, and accuracy. The general scheme of the automatic navigation of the vehicle then can be represented by the sketch in Fig. 2.

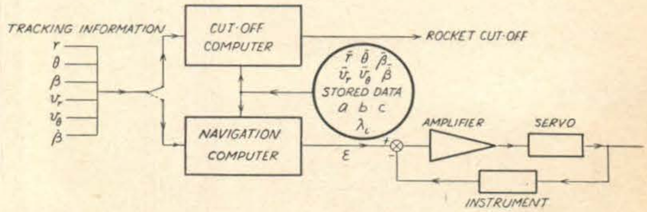

FIG. 2. SYSTEM OF AUTOMATIC NAVIGATION

The primary purpose or the "duty" of the computer is then to properly digest the dynamic and the aerodynamic information of the vehicle and thus to decide the right flight path correction that will insure landing at the chosen destination. The purpose or the duty of the elevator servo is now simply to follow the command of the computer. If the servo is considered to be *internal* to the vehicle, then the computer can be considered to be a mechanism to account for the *external* conditions of the vehicle. This separation of the *internal* from the *external* is not necessary for the control of conventional aircraft. For conventional aircraft, the basic disturbance dynamic equations have constant coefficients and the computer can be merged with the servo through a single "amplifier" of essentially RC-circuits.

The computers envisaged here are carried in the vehicle and receive the information on positions and velocities from the fixed ground tracking stations (4). It will be beyond the scope of the present paper to discuss their design. But the required accuracy and speed to generate proper signal almost instantly would indicate that they should be of the electronic digital type. If this is indeed the proper type to be used, then the separation of cut-off computer and the navigation computer is not necessary. All could be done in one computer by proper programming.

Appendix

Calculation of Coefficients

The quantities F, G, and H are defined by Equation [15]. They contain the parameters Ψ, Λ, Δ, and N.

According to their definition as given by Equations [5] and [14], they can be written as follows:

$$\left.\begin{aligned}
\Psi &= \frac{fg}{W}\\
\Lambda &= \frac{g}{W}\frac{1}{2}\rho S C_L \sqrt{v_r^2 + (v_\theta + w)^2}\\
\Delta &= \frac{g}{W}\frac{1}{2}\rho S C_D \sqrt{v_r^2 + (v_\theta + w)^2}\\
N &= \frac{1}{I}\frac{1}{2}\rho S C_M \{v_r^2 + (v_\theta + w)^2\}
\end{aligned}\right\}\quad\ldots\ldots\ldots[45]$$

where the aerodynamic coefficients C_L, C_D, C_M are functions of angle of attack α, elevator angle ϵ, Mach number M, and the Reynolds number Re. These aerodynamic parameters are related to quantities immediately connected with the flight path as follows:

$$\alpha = \beta - \tan^{-1}\frac{v_r}{v_\theta + w},\quad M = V/a(r),\quad Re = \rho V l/\mu(r) \quad\ldots[46]$$

where $a(r)$ is the sound velocity in atmosphere, and $\mu(r)$ is the coefficient of viscosity of air, both functions of altitude r. In the following calculation, the thrust f will be considered to be a function of altitude only. It is also assumed that the composition of atmosphere at different altitudes remains the same as that of the standard atmosphere, only the density ρ and the temperature T change. Thus the variations of a and μ at any altitude are variations due to temperature T.

For Ψ

$$\frac{\partial\Psi}{\partial r} = \frac{g}{W}\frac{\partial f}{\partial r},\qquad \frac{\partial\Psi}{\partial W} = -\frac{\Psi}{W}\ldots\ldots\ldots\ldots[47]$$

All other partial derivatives are zero.

For Λ

$$\left.\begin{aligned}
\frac{\partial\Lambda}{\partial r} &= \Lambda\Bigg[\frac{1}{\rho}\frac{d\rho}{dr}\left\{1 + \frac{Re}{C_L}\frac{\partial C_L}{\partial Re}\right\} + \frac{1}{V^2}\frac{dw}{dr}\times\\
&\qquad \left\{\left(\frac{M}{C_L}\frac{\partial C_L}{\partial M} + \frac{Re}{C_L}\frac{\partial C_L}{\partial Re} + 1\right)\times (v_\theta + w) + \frac{1}{C_L}\frac{\partial C_L}{\partial\alpha}v_r\right\} -\\
&\qquad \frac{M}{C_L}\frac{\partial C_L}{\partial M}\frac{1}{a}\frac{da}{dr} - \frac{Re}{C_L}\frac{\partial C_L}{\partial Re}\frac{1}{\mu}\frac{d\mu}{dr}\Bigg]\\
\frac{\partial\Lambda}{\partial v_r} &= \Lambda\frac{v_r}{V^2}\left[\frac{M}{C_L}\frac{\partial C_L}{\partial M} + \frac{Re}{C_L}\frac{\partial C_L}{\partial Re} + 1 - \frac{1}{C_L}\frac{\partial C_L}{\partial\alpha}\frac{v_\theta + w}{v_r}\right]\\
\frac{\partial\Lambda}{\partial v_\theta} &= \Lambda\frac{v_\theta + w}{V^2}\left[\frac{M}{C_L}\frac{\partial C_L}{\partial M} + \frac{Re}{C_L}\frac{\partial C_L}{\partial Re} + 1 + \frac{1}{C_L}\frac{\partial C_L}{\partial\alpha}\frac{v_r}{v_\theta + w}\right]\\
\frac{\partial\Lambda}{\partial\beta} &= \Lambda\frac{1}{C_L}\frac{\partial C_L}{\partial\alpha}\\
\frac{\partial\Lambda}{\partial\epsilon} &= \Lambda\frac{1}{C_L}\frac{\partial C_L}{\partial\epsilon}\\
\frac{\partial\Lambda}{\partial\rho} &= \Lambda\frac{1}{\rho}\left[1 + \frac{Re}{C_L}\frac{\partial C_L}{\partial Re}\right]\\
\frac{\partial\Lambda}{\partial T} &= -\Lambda\left[\frac{M}{C_L}\frac{\partial C_L}{\partial M}\frac{1}{2T} + \frac{Re}{C_L}\frac{\partial C_L}{\partial Re}\frac{1}{\mu}\frac{\partial\mu}{\partial T}\right]\\
\frac{\partial\Lambda}{\partial w} &= \Lambda\frac{v_\theta + w}{V^2}\left[\frac{M}{C_L}\frac{\partial C_L}{\partial M} + \frac{Re}{C_L}\frac{\partial C_L}{\partial Re} + 1 + \frac{1}{C_L}\frac{\partial C_L}{\partial\alpha}\frac{v_r}{v_\theta + w}\right] = \frac{\partial\Lambda}{\partial v_\theta}\\
\frac{\partial\Lambda}{\Lambda W} &= -\frac{\Lambda}{W}
\end{aligned}\right\}\quad\ldots[48]$$

For Δ, the partial derivatives are obtained from the above equations by substituting Δ for Λ, and C_D for C_L.

For N,

$$\left.\begin{aligned}
\frac{\partial N}{\partial r} &= N\Bigg[\frac{1}{\rho}\frac{d\rho}{dr}\left\{1 + \frac{Re}{C_M}\frac{\partial C_M}{\partial Re}\right\} + \frac{1}{V^2}\frac{dw}{dr}\times\\
&\qquad \left\{\left(\frac{M}{C_M}\frac{\partial C_M}{\partial M} + \frac{Re}{C_M}\frac{\partial C_M}{\partial Re} + 2\right)\times (v_\theta + w) + \frac{1}{C_M}\frac{\partial C_M}{\partial\alpha}v_r\right\} -\\
&\qquad \frac{M}{C_M}\frac{\partial C_M}{\partial M}\frac{1}{a}\frac{da}{dr} - \frac{Re}{C_M}\frac{\partial C_M}{\partial Re}\frac{1}{\mu}\frac{d\mu}{dr}\Bigg]\\
\frac{\partial N}{\partial v_r} &= N\frac{v_r}{V^2}\left[\frac{M}{C_M}\frac{\partial C_M}{\partial M} + \frac{Re}{C_M}\frac{\partial C_M}{\partial Re} + 2 - \frac{1}{C_M}\frac{\partial C_M}{\partial\alpha}\frac{v_\theta + w}{v_r}\right]\\
\frac{\partial N}{\partial v_\theta} &= N\frac{(v_\theta + w)}{V^2}\left[\frac{M}{C_M}\frac{\partial C_M}{\partial M} + \frac{Re}{C_M}\frac{\partial C_M}{\partial Re} + 2 + \frac{1}{C_M}\frac{\partial C_M}{\partial\alpha}\frac{v_r}{v_\theta + w}\right]\\
\frac{\partial N}{\partial\beta} &= N\frac{1}{C_M}\frac{\partial C_M}{\partial\alpha}\\
\frac{\partial N}{\partial\epsilon} &= N\frac{1}{C_M}\frac{\partial C_M}{\partial\epsilon}\\
\frac{\partial N}{\partial\rho} &= N\frac{1}{\rho}\left[1 + \frac{Re}{C_M}\frac{\partial C_M}{\partial Re}\right]\\
\frac{\partial N}{\partial T} &= -N\left[\frac{M}{C_M}\frac{\partial C_M}{\partial M}\frac{1}{2T} + \frac{Re}{C_M}\frac{\partial C_M}{\partial Re}\frac{1}{\mu}\frac{\partial\mu}{\partial T}\right]\\
\frac{\partial N}{\partial w} &= \frac{\partial N}{\partial v_\theta}\\
\frac{\partial N}{\partial I} &= -\frac{N}{I}
\end{aligned}\right\}\quad\ldots[49]$$

With these partial derivatives, the coefficients a's, b's, and c's can be easily calculated:

$$\left.\begin{aligned}
a_1 &= \frac{\partial F}{\partial r} = \frac{\partial\Psi}{\partial r}\sin\beta + \frac{dw}{dr}\Lambda + (v_\theta + w)\frac{\partial\Lambda}{\partial r} - v_r\frac{\partial\Delta}{\partial r} + \left(\frac{v_\theta}{r} \pm \Omega\right)^2 - 2\frac{v_\theta}{r}\left(\frac{v_\theta}{r} \pm \Omega\right) + 2\frac{g}{r}\left(\frac{r_0}{r}\right)^2\\
a_2 &= \frac{\partial F}{\partial\beta} = \Psi\cos\beta + (v_\theta + w)\frac{\partial\Lambda}{\partial\beta} - v_r\frac{\partial\Delta}{\partial\beta}\\
a_3 &= \frac{\partial F}{\partial v_r} = (v_\theta + w)\frac{\partial\Lambda}{\partial v_r} - \Delta - v_r\frac{\partial\Delta}{\partial v_r}\\
a_4 &= \frac{\partial F}{\partial v_\theta} = \Lambda + (v_\theta + w)\frac{\partial\Lambda}{\partial v_\theta} - v_r\frac{\partial\Delta}{\partial v_\theta} + 2\left(\frac{v_\theta}{r} \pm \Omega\right)\\
a_5 &= \frac{\partial F}{\partial\epsilon} = (v_\theta + w)\frac{\partial\Lambda}{\partial\epsilon} - v_r\frac{\partial\Delta}{\partial\epsilon}\\
a_6 &= \frac{\partial F}{\partial\rho} = (v_\theta + w)\frac{\partial\Lambda}{\partial\rho} - v_r\frac{\partial\Delta}{\partial\rho}\\
a_7 &= \frac{\partial F}{\partial T} = (v_\theta + w)\frac{\partial\Lambda}{\partial T} - v_r\frac{\partial\Delta}{\partial T}\\
a_8 &= \frac{\partial F}{\partial w} = \Lambda + (v_\theta + w)\frac{\partial\Lambda}{\partial w} - v_r\frac{\partial\Delta}{\partial w}\\
a_9 &= \frac{\partial F}{\partial W} = \frac{\partial\Psi}{\partial W}\sin\beta + (v_\theta + w)\frac{\partial\Lambda}{\partial W} - v_r\frac{\partial\Delta}{\partial W}
\end{aligned}\right\}\quad\ldots[50]$$

$$\left.\begin{aligned}
b_1 &= \frac{\partial G}{\partial r} = \frac{\partial\Psi}{\partial r}\cos\beta - v_r\frac{\partial\Delta}{\partial r} - \frac{dw}{dr}\Delta - (v_\theta + w)\frac{\partial\Delta}{\partial r} + \frac{v_r v_\theta}{r^2}\\
b_2 &= \frac{\partial G}{\partial\beta} = -\Psi\sin\beta - v_r\frac{\partial\Lambda}{\partial\beta} - (v_\theta + w)\frac{\partial\Delta}{\partial\beta}
\end{aligned}\right.\quad\ldots[51]$$

$$
\left.\begin{aligned}
b_3 &= \frac{\partial G}{\partial v_r} = -\Lambda - v_r \frac{\partial \Lambda}{\partial v_r} - (v_\theta + w)\frac{\partial \Delta}{\partial v_r} - \\
&\qquad 2\left(\frac{1}{2}\frac{v_\theta}{r} \pm \Omega\right) \\
b_4 &= \frac{\partial G}{\partial v_\theta} = -v_r \frac{\partial \Lambda}{\partial v_\theta} - \Delta - (v_\theta + w)\frac{\partial \Delta}{\partial v_\theta} - \frac{v_r}{r} \\
b_5 &= \frac{\partial G}{\partial \epsilon} = -v_r \frac{\partial \Lambda}{\partial \epsilon} - (v_\theta + w)\frac{\partial \Delta}{\partial \epsilon} \\
b_6 &= \frac{\partial G}{\partial \rho} = -v_r \frac{\partial \Lambda}{\partial \rho} - (v_\theta + w)\frac{\partial \Delta}{\partial \rho} \\
b_7 &= \frac{\partial G}{\partial T} = -v_r \frac{\partial \Lambda}{\partial T} - (v_\theta + w)\frac{\partial \Delta}{\partial T} \\
b_8 &= \frac{\partial G}{\partial w} = -v_r \frac{\partial \Lambda}{\partial w} - \Delta - (v_\theta + w)\frac{\partial \Delta}{\partial w} \\
b_9 &= \frac{\partial G}{\partial W} = \frac{\partial \Psi}{\partial W}\cos\beta - v_r \frac{\partial \Lambda}{\partial W} - (v_\theta + w)\frac{\partial \Delta}{\partial W}
\end{aligned}\right\} \quad ..[51] \ (cont'd)
$$

$$
\left.\begin{aligned}
c_1 &= \frac{\partial H}{\partial r} = -\frac{1}{r^2}\left\{\Psi\cos\beta - v_r\Lambda - (v_\theta + w)\Delta - \right. \\
&\qquad \left. 2v_r\left(\frac{v_\theta}{r} \pm \Omega\right)\right\} + \frac{1}{r}\left\{\frac{\partial \Psi}{\partial r}\cos\beta - v_r\frac{\partial \Lambda}{\partial r} - \right. \\
&\qquad \left. (v_\theta + w)\frac{\partial \Delta}{\partial r} - \frac{dw}{dr}\Delta + 2\frac{v_r v_\theta}{r^2}\right\} + \frac{\partial N}{\partial r} \\
c_2 &= \frac{\partial H}{\partial \beta} = \frac{1}{r}\left\{-\Psi\sin\beta - v_r\frac{\partial \Lambda}{\partial \beta} - \right. \\
&\qquad \left. (v_\theta + w)\frac{\partial \Delta}{\partial \beta}\right\} + \frac{\partial N}{\partial \beta} \\
c_3 &= \frac{\partial H}{\partial v_r} = \frac{1}{r}\left\{-\Lambda - v_r\frac{\partial \Lambda}{\partial v_r} - (v_\theta + w)\frac{\partial \Delta}{\partial v_r} - \right. \\
&\qquad \left. 2\left(\frac{v_\theta}{r} \pm \Omega\right)\right\} + \frac{\partial N}{\partial v_r} \\
c_4 &= \frac{\partial H}{\partial v_\theta} = \frac{1}{r}\left\{-v_r\frac{\partial \Lambda}{\partial v_\theta} - \Delta - (v_\theta + w)\frac{\partial \Delta}{\partial v_\theta} - \right. \\
&\qquad \left. 2\frac{v_r}{r}\right\} + \frac{\partial N}{\partial v_\theta} \\
c_5 &= \frac{\partial H}{\partial \epsilon} = \frac{1}{r}\left\{-v_r\frac{\partial \Lambda}{\partial \epsilon} - (v_\theta + w)\frac{\partial \Delta}{\partial \epsilon}\right\} + \frac{\partial N}{\partial \epsilon} \\
c_6 &= \frac{\partial H}{\partial \rho} = \frac{1}{r}\left\{-v_r\frac{\partial \Lambda}{\partial \rho} - (v_\theta + w)\frac{\partial \Delta}{\partial \rho}\right\} + \frac{\partial N}{\partial \rho} \\
c_7 &= \frac{\partial H}{\partial T} = \frac{1}{r}\left\{-v_r\frac{\partial \Lambda}{\partial T} - (v_\theta + w)\frac{\partial \Delta}{\partial T}\right\} + \frac{\partial N}{\partial T} \\
c_8 &= \frac{\partial H}{\partial w} = \frac{1}{r}\left\{-v_r\frac{\partial \Lambda}{\partial w} - \Delta - (v_\theta + w)\frac{\partial \Delta}{\partial w}\right\} + \frac{\partial N}{\partial w} \\
c_9 &= \frac{\partial H}{\partial W} = \frac{1}{r}\left\{\frac{\partial \Psi}{\partial W}\cos\beta - v_r\frac{\partial \Lambda}{\partial W} - (v_\theta + w)\frac{\partial \Delta}{\partial W}\right. \\
c_{10} &= \frac{\partial H}{\partial I} = \frac{\partial N}{\partial I}
\end{aligned}\right\} \quad ..[52]
$$

After the power cut-off, the thrust f vanishes. Thus for $t > \bar{t}_1$, Ψ and its derivatives are zero.

References

1 "Mathematical Theory of Rocket Flight," by J. B. Rosser, R. R. Newton, and G. L. Gross, McGraw Hill Book Co., Inc., New York, 1947, particularly chapter III.

2 "The Perturbation Calculus in Missile Ballistics," by R. Drenick, *Journal of the Franklin Institute,* vol. 251, 1951, pp. 423–436.

3 "Mathematics for Exterior Ballistics," by G. A. Bliss, John Wiley & Sons, New York, 1944.

4 For possible tracking system, see "Pictorial Radio" by C. D. Tuska, *Journal of the Franklin Institute,* vol. 253, 1952, pp. 1–20; pp. 95–124.

13. Control Design by Specified Criteria

In the previous chapters, we have considered the problem of control for systems of increasing generality. But we have so far limited ourselves to the linear problem. A very general approach to the problem of control is to consider it as a system which has a free variable at the disposal of the designer. The problem of the designer is to choose a relation between this free variable and the other variables such that the other variables of the system satisfy a certain control criteria specified by the problem. For instance, the problem of servo-stabilization of the combustion in the rocket motor can be considered as a dynamic system between the propellant injection rate, the chamber pressure and the capacity of the control capacitance in the feed line. There are two differential equations relating the three variables. The problem is to choose another relation between the three variables so that the overall system satisfies the condition of stability.

In this chapter, we shall treat the problem of control and stability from this general point of view, applicable to either linear or non-linear systems. Parts of the discussion follow the paper by A. S. Boksenbom and R. Hood.*

Control Problem

An important aspect of the control-synthesis problem is a clear definition of the criterions of desired control behavior. If a variable y is to be controlled, a reasonable criterion is that the time integral

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* A. S. Boksenbom and R. Hood, "Automatic Control Systems Satisfying certain General Criterions on Transient Behavior" NACA TN 2378 (1951).

12. Optimalizing Control

In the previous chapters we have successively discussed control systems with increasing degrees of generality. However, one basic characteristic of these control systems remain the same: The properties or the performances of the system to be controlled are always assumed to be known. In fact, the control design is based upon these known properties of the controlled system. The success and the accuracy of the control are thus dependent upon how closely the actual properties of the controlled system are predicted. Since the performance of all engineering systems are subject to small but appreciable variations with time, extreme accuracy of controlled behavior cannot be expected from the previously discussed designs of controls. In this chapter we shall discuss the optimalizing control, first analysed by C. S. Draper, Y. T. Li and H. Laning, Jr. This control is a simple example of continuous sensing and measuring control system where no exact knowledge of the properties of the controlled system is necessary for the successful design of the control. Our discussion here follows closely a paper by Y. T. Li.*

Basic Concept

An optimalizing controller is designed to force the output of the operating system to remain within a tolerable deviation from the optimum output level, which is not readily known. An operating system in a industrial plant generally functions to produce a certain type of output from a number of inputs. Both the output and the essential inputs may be in the form of certain types of material or certain forms of power. The production or consumptuon of these materials or this power, is the primary concern in the operation. The internal combustion engine is an example of such an operating system. In this system, the

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* "Opeimalizing System for Process Control", Instruments, Vol. 25, Nos. 1, 2, 3; pp. 72-77, 190-193, 228, 324-327, 350-352, (1952). "Principles of Optimalizing Control Systems and an Application to the Internal Combustion Engine" by C. S. Draper and Y. T. Li, ASME Publications, p. 160 (1951).

fuel may be considered as the first actuating input, air as the second actuating input. The ignition timing can be considered as the modifying input which is non-consuming but changes the functional relationship between the actuating inputs and the output. The combustion chamber and swept-cylinder volume provide the input converter function by supplying an enclosure in which chemical energy is changed, first into heat and then into mechanical output. The piston, connecting rod, crankshaft, and drive shaft act as the output coupler that serves to transfer the mechanical energy output of the cylinder gases into the mechanical work of rotation which is supplied to a brake or electrical generator that acts as the output-absorbing system.

When a given load requirement is imposed on a system like that of an internal combustion engine, one of the actuating inputs must be adjusted to a level determined principally by the load. In order for operation to be possible at all, the other actuating inputs and modifying input must have more or less exact relationships to this primary controlled input. The adjustments of these inputs in order to fulfill the given load condition can be made by suitable performance regulating systems. In the case of the internal combustion engine, a governor and a carburetor may be used together when the load requirement is to maintain the speed of the engine at a constant level.

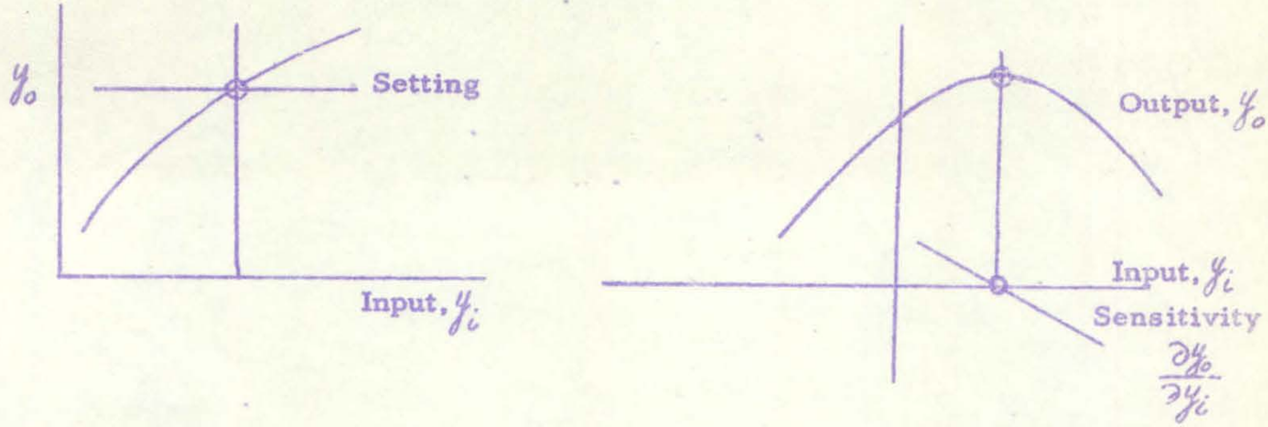
When the input settings are made within ranges that allow the system to function, the remaining problem is to refine the input adjustments so that the required output is obtained with minimum flow rates of the costly inputs, or the maximum output is obtained for given flow rates of these inputs. This is the optimum point of operation. The purpose of an optimalizing control is to force the controlled system to operate as closely as possible to this optimum point. This searching of the optimum point is done by utilizing the non-linear nature of the performance characteristics between the output and the controlled inputs. In this type of system the operation is independent of all the complicated relationships of the controlled input to the load conditions and the environmental inputs. For this reason, for complicated systems, the optimalizing control has an advantage over the conventional control not only in accuracy but quite possibly also in simplicity and in the directness of approach to the problem.

In fact, optimum performance through optimalizing control can frequently be accomplished by manual means. An operator, with the aid of a suitable output indicating system such as a torque meter, may progressively adjust the ignition timing and fuel-air ratio to proper settings for optimum output. In principle this process, in which the operator makes munual adjustments, is a feedback operation. the operator actually correlates the information of the torque meter with the input adjustments he has made to determine the additional adjustment required. Humanly-controlled operating systems are necessarily slow in response, and they are generally not satisfactory, or even practicable in many operations.

An optimalizing control system operates in principle similar to the manual adjustments described above. However, in the optimalizing control system an automatic controller is used in the feedback loop to replace the human operator. The observations and the correlation of the output with the controlled input adjustment, and the cut-and-try process exercised by the human operator, are all included as functions of the controller.

General Principles of Optimalizing Control

Most types of controllers are actuated by the output deviations of the controlled system. This quantity is by definition the difference between the actual level of the controlled output and some setting for The main action of the controller is to reduce the output deviation to zero. The optimalizing controller is different from this in that the



purpose is to operate at the optimum point or point of vanishing "sensitivity", $\partial y_o / \partial y_i$. Therefore here the sensitivty takes the role of the output deviation.

However, the basic concept of optimalizing control is the actual sensing of the optimum point during operation. This means that the principal problem in the design of a controller using output deviations as its essential input is to generate an indicated optimum signal which is a sufficiently close representation of the actual optimum output level as this quantity varies with controlled system variation. When controlled system sensitivity signals are used for control purposes, the reference level is naturally zero and for this reason is simple to realize in practice. On the other hand, the problem of generating signals that satisfactorily represent controlled system sensitivity is not simple. The several possible approaches to this problem lead to various controller types.

There are two classes of optimalizing controllers, depending upon the type of input signal that is used for operation. These classes, with sub-classes, are as follows:

1. Input-output sensitivity operated controllers
    a) Sensitivity signal input controllers
    b) Continuous test signal controllers
    c) Output sampling controllers
2. Peak holding controllers

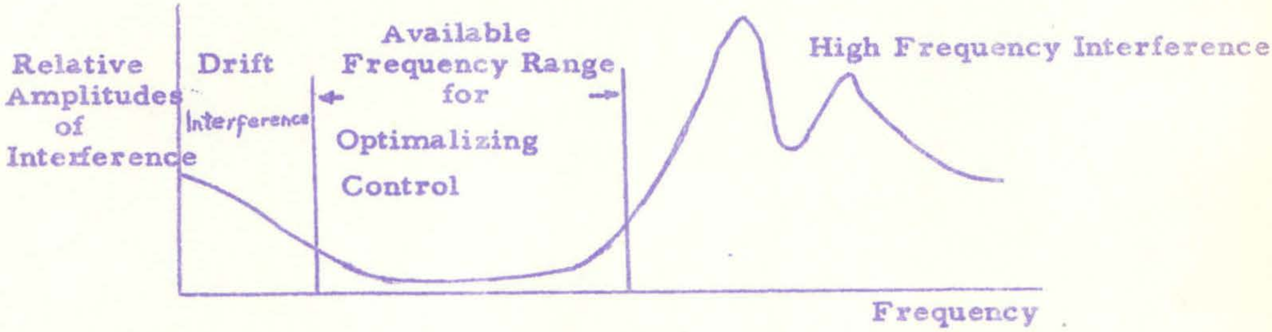
All optimalizing controllers of the first type act by using the controlled system itself to generate a signal which represents its controlled input-output sensitivity and then using this signal as the essential controller input signal. Optimalizing controllers of the second type operate by continuously searching for an indicated optimum output level signal which is used in the generation of the indicated output deviation signal. We shall discuss these various controllers in the following sections.

Limitation of Optimalizing Control Test Functions by Rand on Interference

To generate a sensitivity signal or to establish an indicated optimum level, data from more than one operating point are necessary. In practice some test variation of a suitable function with time is applied to the controlled input. The effect of the test input variation on the output is to produce a corresponding output variation. By proper correlation of the test input variation with the corresponding output variation it is sometimes possible to determine the input-output sensitivity or the indicated optimum output level.

In the vicinity of the optimum level, the effect of the input test variation is to produce an output variation which in general remains below the optimum output level within an output hunting zone. This output hunting action is responsible for some output hunting loss and represents the cost of using optimalizing control. It is generally desirable to keep the hunting zone as small as possible by using the smallest practicable input test variation.

In order to measure effectively the output variation as a result of the input test variation, it is necessary for this output variation with time to be made up of frequency components that may with certainty be distinguished from output variations due to interference effects. The relative amplitudes of the output frequency components making up the interference spectrum for a typical operating system is given in the following figure. Drift interferences are associated



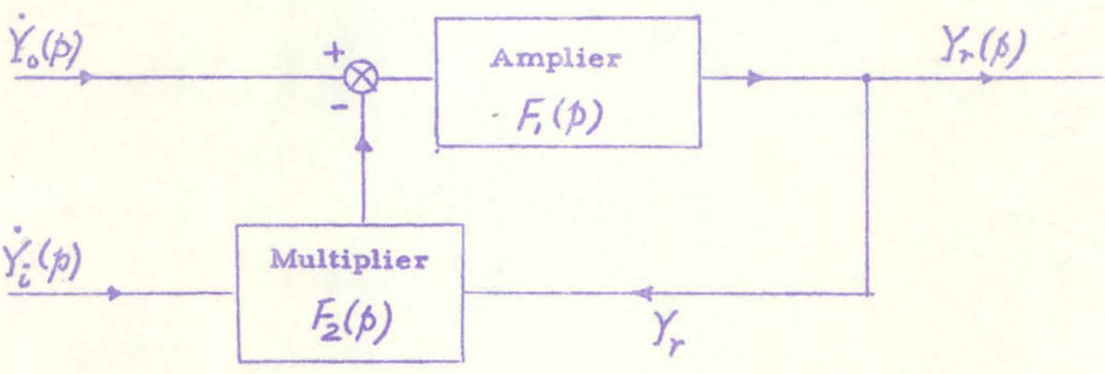
with slow changes in the environment or internal conditions of the operating system and for this reason are made up of relatively low frequencies. High-frequency interference occurs above some more or less sharply defined lower limit. Satisfactory optimalizing control depends upon the existence of some frequency range, substantially free from strong interference components, that is between the low- and high- frequency ranges. The corresponding control problem is to design equipment based upon test functions formed of frequency components that may be recognized in the presence of normal output interference without causing unacceptably large hunting loss. In general, the test function must be made up of input variations that are fast enough to be distinguishable from drift interference, and at the same time are slow enough to prevent confusion with high-frequency interference effects.

Sensitivity Signal Input Controller

We shall first discuss the various possible forms of the optimalizing controller without consideration of the dynamic and the damping effects of the various components. This is done in order to bring the relation between the controlled system and the controller into its clearest possible view. The dynamic effects will be added later.

In principle, the input-output sensitivity of a controlled system can be obtained by introducing a suitable input variation to the controlled system, and dividing the rate of change of the output signal by the rate of change of the input signal of the test variation. The process of taking the ratio between the output rate signal and the input rate eliminates the time element from the function, and the resulting quotient is virtually the sensitivity signal of the controlled system.

As an example of a simple circuit for ratio taking, consider the following diagram:



Let $Y_o(p)$ be the Laplace transform of the output rate signal, be the Laplace transform of the input rate signal, $Y_i(p)$ the Laplace transform of the ratio signal, and $F_2(p)$, $F_2(p)$ the transfer functions, then

$$Y_r = F_1(p)\left[\dot{Y}_o(p) - F_2(p)\,\dot{Y}_i(p)\,Y_r(p)\right]$$

Or

$$Y_r = \frac{F_1(p)\,\dot{Y}_o(p)}{1 + F_1(p)F_2(p)\dot{Y}_i(p)} \qquad (11.1)$$

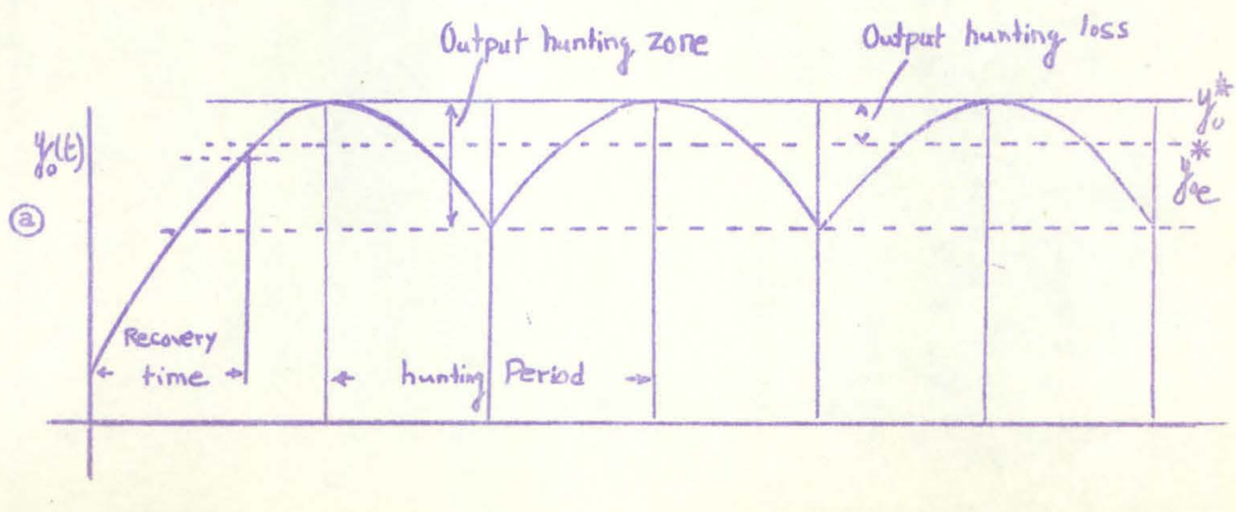
Therefore when the gain of the amplier is very large,

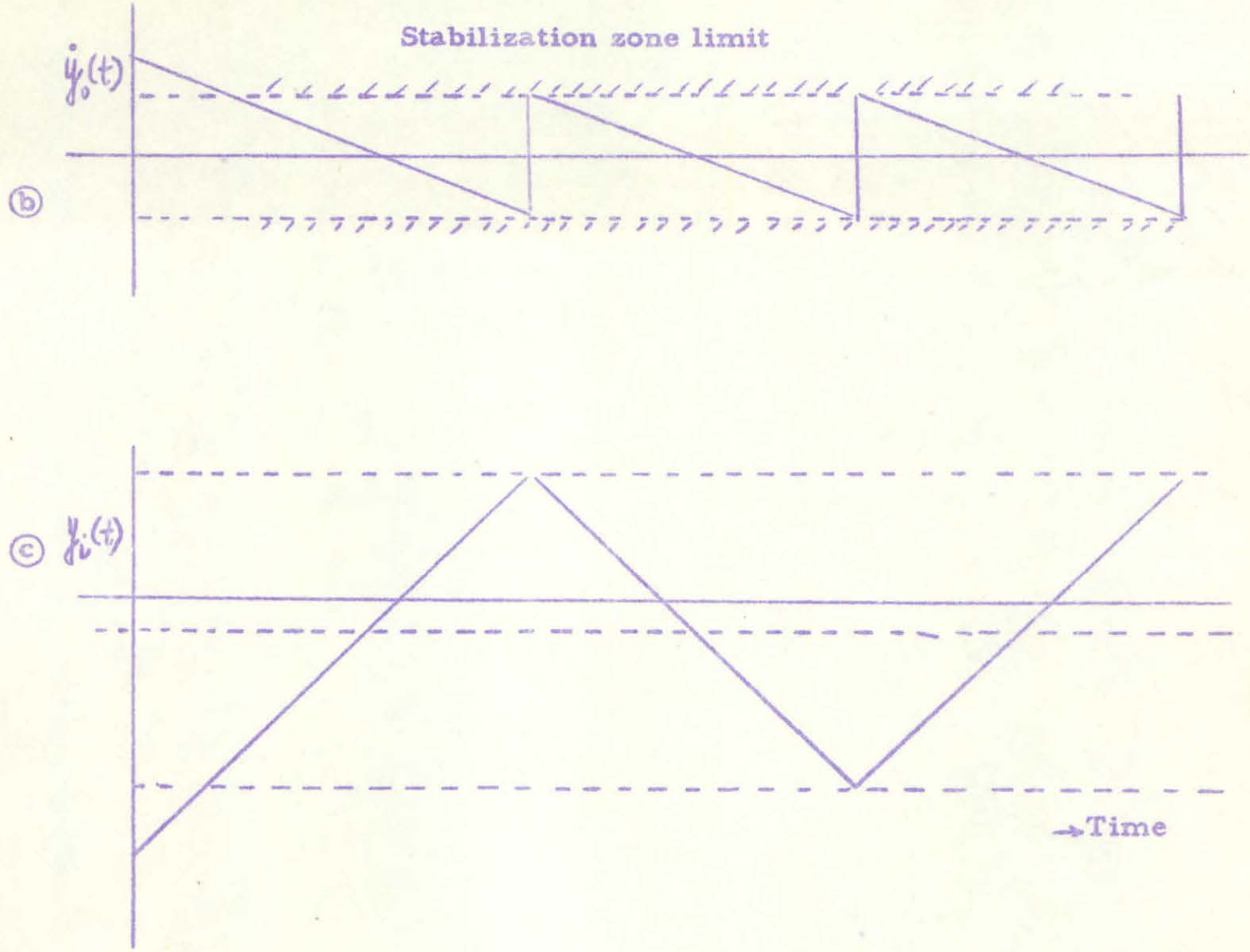
$$Y_r \cong \frac{1}{F_2(p)}\left[\frac{\dot{Y}_o(p)}{\dot{Y}_i(p)}\right] \qquad (11.2)$$

Thus is approximately proportional to the ratio of output rate signal and the input rate signal, or

$$y_r = \frac{dy_o/dt}{dy_i/dt} = \left(\frac{dy_o}{dy_i}\right) \qquad (11.3)$$

For simplicity in illustration, an input variation with a constant rate which may be reversed in direction as required for control purposes is considered in the discussion. Under this condition the output rate signal may be used to represent the sensitivity of the controlled system, with a sign that depends upon the direction of the input variation.





In the accompanying diagram, (a), (b) and (c) are respectively the controlled output, $y_o(t)$, the output rate, $\dot{y}_o(t)$ and the controlled input $y_i(t)$. The controlled input variation with time is generated by a constant speed change of input with reversal at points determined by operation of the controller. Under this situation the output rate signal produced by the input correction signal is equivalent to the controlled system sensitivity. So long as this output rate signal is larger than a stabilization zone limit, the system is designed so that the controlled input adjustor drive will cause the controlled input to vary in either direction at a constant rate. This action is illustrated in the figure. When the output signal curve has passed the optimum point and started to decrease, the output rate signal is allowed to become negative until it has passed through the stabilization zone. At the lower limit of this zone the direction of the controlled input change is reversed. The size of the stabilization zone is chosen so that the controller will not respond to interference components that may be present in the output signal.

The operating cycle described in the preceding paragraph continues indefinitely. The result of this action is that the automatically controlled triangular input variation change produces a repeating series of concave output signal curve segments. The output hunting zone is by definition the difference between the highest and lowest output levels that exist under the action of the controller. Because this hunting zone exists, the effective output, which is by definition the constant output level that would be required to produce the same average output effect as that actually produced by the controlled system with the controller operating, is less than the optimum output level. The difference between the optimum output and the effective output is the output hunting loss.

In many practical situations the controlled input-output characteristic near the optimum point may be approximated well by a parabolic curve. When this approximation is valid for a controlled system, it may be shown by analysis that the output hunting loss is 1/3 of the hunting zone. This means that one of the prime objectives in optimalizing controller design is to reduce the output hunting zone to the lowest feasible magnitude.

Another performance parameter is the output recovery time. It is defined as the time required for the controlled system to eliminate 95 $^{o}/o$ of the required output correction from an arbitrary initial setting which is different from the final effective output level. The choice of 95 $^{o}/o$, instead of the complete initial correction, for the definition of recovery time simplifies measurements by allowing the terminal instant to be taken when the rate of change of output is still large enough for accurate observation. In addition, if the controlled system operation follows an approximately exponential law, the recovery time is equal to 3 characteristic time of the exponential ($\log_e 20 \approx 3$).

In practice, controlled systems are often subjected to sudden changes in operating conditions. To minimize output losses during the transient periods following changes, and to insure generally stable operation, short recovery times are necessary for satisfactory performance of the controlled system.

The direct sensitivity optimalizing controller described represents the simplest design for an optimalizing controller. For this reason it has been used to illustrate the concepts associated with controller performance and to establish the requirements for operation, although superior results may be achieved with other arrangements. The primary difficulty in a direct sensitivity optimalizing controller is that the control signal is produced by differentiation of a signal representing the output. This differentiation process acts to increase the effects of high-frequency interference components so that relatively large stabilization zones must be used. For controlled systems with small amplitude high-frequency interference components direct sensitivity control may operate properly. On the other hand, the interference components present in the output of a piston type internal combustion engine are so large that the necessary sensitivity stabilization zone does not permit close output control.

Continuous Test Signal Controller

In a continuous test signal controller the controlled system sensitivity is obtained by correlating the output variation of the controlled system with an input test variation of a constant sinusoidal function. It is mentioned in the previous section that the sensitivity of the controlled system can be represented by the ratio of the output rate to the input rate. When a sinusoidal test variation is used, the controlled system sensitivity is proportional to the average of the product of the sinusoidal input test signal and the corresponding output test signal. To show this, let y_o^* be the optimum output, accuring at the input y_i^*. Then for a parabolic variation of y_o against y_i .

$$y_o = y_o^* - K(y_i - y_i^*)^2 \tag{11.4}$$

Now if the test function is a simusoidal variation around the operating input level y_i , i.e.,

$$y_i = y_i^o + a\sin\omega t \tag{11.5}$$

Then the corresponding output is

$$\begin{aligned} y_o(t) &= y_o^* - K\left[(y_i^o - y_i^*) + a\sin\omega t\right]^2 \\ &= y_o^* - K(y_i^o - y_i^*)^2 - 2Ka(y_i^o - y_i^*)\sin\omega t - Ka^2\sin^2\omega t \end{aligned} \tag{11.6}$$

But the output at the operating input y_i^o is y_o^o given by

$$y_o^o = y_o^* - K(y_i^o - y_i^*)^2 \tag{11.7}$$

The difference of Eqs. (11.6) and (11.7) gives the output test signal $\Delta y_o(t)$,

$$\Delta y_o(t) = -2Ka(y_i^o - y_i^*)\sin\omega t - Ka^2\sin^2\omega t \tag{11.8}$$

The input test signal $\Delta y_i(t)$ is $a\sin\omega t$, therefore the product of the test signals is

$$\Delta y_o(t) \cdot \Delta y_i(t) = -2Ka^2(y_i^o - y_i^*)\sin^2\omega t - Ka^3\sin^3\omega t$$

The time average is then

$$\overline{\Delta y_o(t) \cdot \Delta y_i(t)} = -2\frac{\pi K}{\omega}a^2(y_i^o - y_i^*) = \frac{\pi a^2}{\omega}\left(\frac{dy_o}{dy_i}\right)_{y_i^o} \tag{11.9}$$

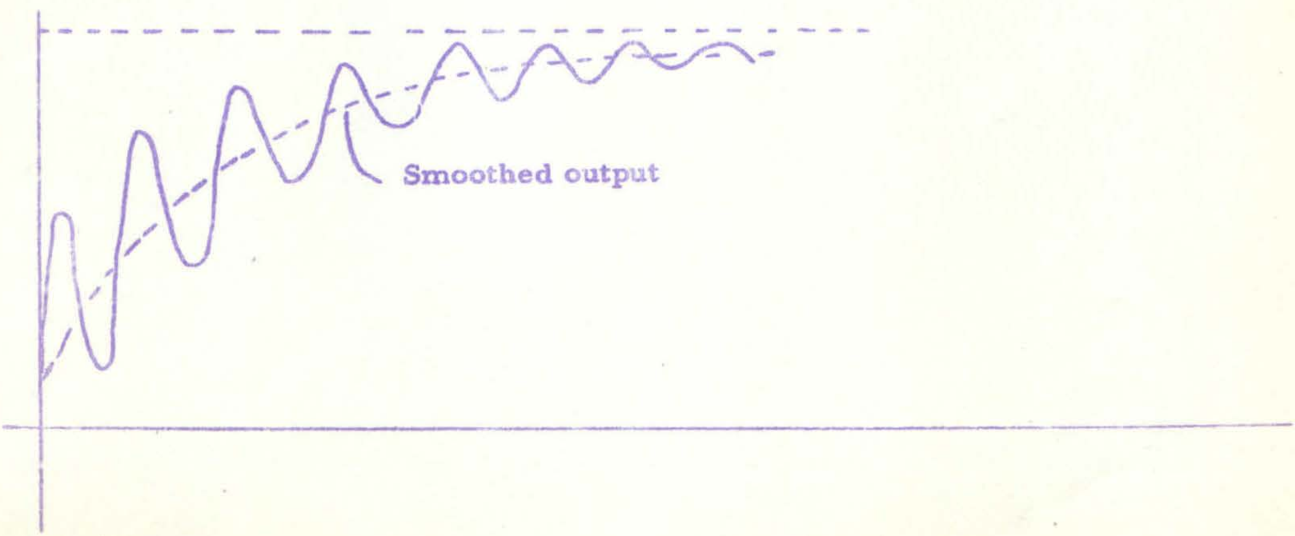
Actually the process of forming the product and taking the average is accomplished in several steps. The output signal is supplied to the band-pass filter, which separates the fundamental component of the test signal output from the smoothed output signal y_o^o and suppresses both high-frequency components of the test signal output and high frequency interference components. The output of the band-pass filter is then the first term to the right of Eq. (11.8). This fundamental frequency test signal component then passes through the phase adjustor, which compensates for undesirable phase shift effect that may have been introduced by operations of the controlled system. The phase-adjusted test signal is supplied as in input to the reactifying multiplier, which also receives the input test signal. The rectifying multiplier in effect takes the product of these two signals and gives the resultant an algebraic sign by using phase-sensitive rectification of the output signal component with respect to the input test signal. In order to prepare this instantaneously varying signal for use as the controlling effect for an input adjustor drive system, it is passed through the smoothing filter, which produces the input correction signal. The input correction signal then controls the rate of input variation.

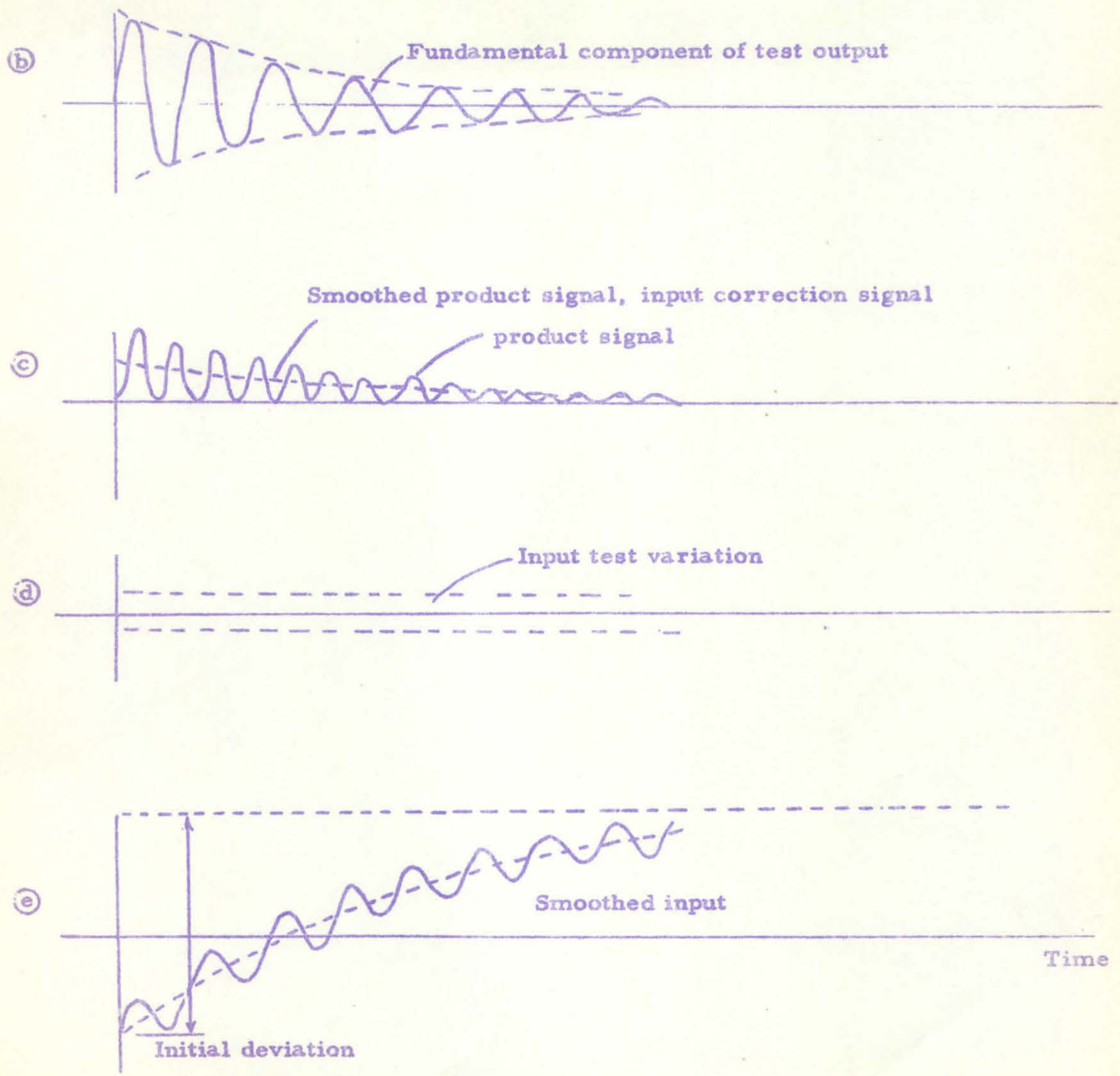
Since the input correction signal is essentially perportional to the sensitivity, the control system described will approach asymptotically to the point of zero sensitivity, or the optimum point, with a certain characteristic time.

Since at any time the output deviation is proportional to the square of the input deviation, the smoothed output signal is proportional to the square of the corresponding value of the average input level. Because the average input curve is an exponential function, it is seen that the smoothed output curve is also an exponential function of time, but with an characteristic time 1/2 of that for the input variation.

Continuous test signal optimalizing controllers are generally more satisfactory than controllers of sensitivity signal input type. Firstly, in a continuous test signal optimalizing controller the sensitivity signal is obtained without the use of differentiation with respect to time, which reduces the effects of high-frequency interference. The continuous test signal controller also has the advantage of being able to generate an input correction signal independently of the input adjustor drive speed. Because of this characteristic it is possible to simultaneously produce correction signals for several inputs without mutual interference, if proper separation of test function frequencies are achieved. By using the required number of optimalizing controllers

(a)





(b)
Fundamental component of test output
Smoothed product signal, input correction signal
(c)
product signal
Input test variation
(d)
(e)
Smoothed input
Time
Initial deviation

in parallel, the necessary adjustments to several inputs may be made to drive the controlled system toward the optimum point with minimum loss of time.

The feasibility of the continuous test signal optimalizing controller system depends upon the frequency range of tolerable interference. When a number of inputs are to be controlled, a wide range of frequency of tolerable interference is required. For a single controlled input, the frequency range of tolerable interference should be wide enough to include both the frequency of the test signal and that required for the average input level changes of the input adjustor drive. Controlled systems with a narrow tolerable frequency band are difficult to control with continuous test signal controllers.

A basic difficulty with the exponential type of system response is that the input and output rates become very small as the deviations approach zero. With small input and output rates, system operation tends to become unsatisfactory becuase of interference effects. This trouble can be overcome by reverting to constant rate input testing function method described in the previous section when the deviations are reduced to small values.

Optimalizing Controller with Output Sampling

An output sampling type controller is operated by an output rate signal that is generated by taking differences at discrete time instants between the average output signal levels for these successive equal sampling periods. This output rate signal is used to produce the input correction signal, which is then used to control the hunting operation of an optimalizing system. When reversible constant input rate is used, the operation of the output sampling type optimalizing control system is in general similar to that described as the sensitivity signal input controller.

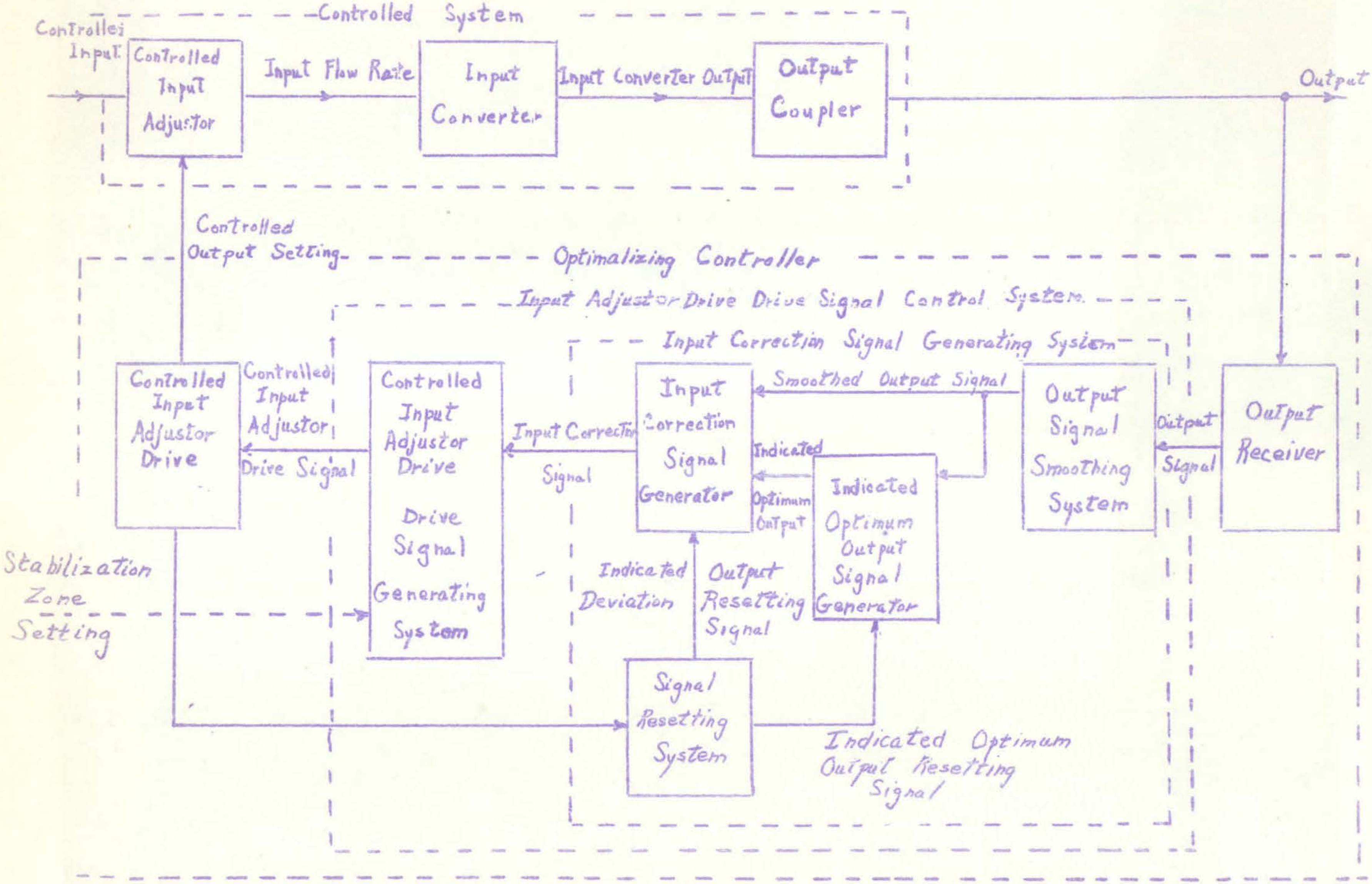
The controlled input of the output sampling type optimalizing control system may also be adjusted to have a rate proportional to the controlled system sensitivity. The result of this type of adjustment is an exponential output recovery curve similar to that shown the continuous test function systems. We shall not discuss the details of this type of optimalizing system here for its similarity to types already shown.

Peak Holding Type of Optimalizing Control System

Optimalizing control systems of the peak holding type differ from the systems already discribed in the method used for generating the input correction signal, which has in general been a representation of the controlling system sensitivity. In systems of the peak holding type, the input correction signal is based on the difference between an indicated optimum output signal as the reference quantity and the output signal as the compared quantity.

In principle the indicated optimum output signal is equal to the output signal so long as the magnitude of this signal is increasing, but remains constant at the highest level reached when the output signal starts to decrease. The operation of the controller is to generate a suitable input correction signal as soon as the output signal starts to decrease, and use this input correction signal to adjust the input to restore the output.

The accompanying figure is a functional diagram for a typical optimalizing control system of the peak holding type. The corresponding performance characteristics are illustrated in a second figure. The controlled input adjustor drive operates to change the input at a constant rate from some typical level. The constant input rate, operating through the characteristic curve, causes the output signal to have the shape illustrated. So long as the output signal is increasing, the indicated optimum output signal changes with the signal. When the magnitude of the output signal passes its peak, which corresponds to the optimum output level, and starts to decrease the indicated optimum output signal is held at its highest level by the indicated optimum output

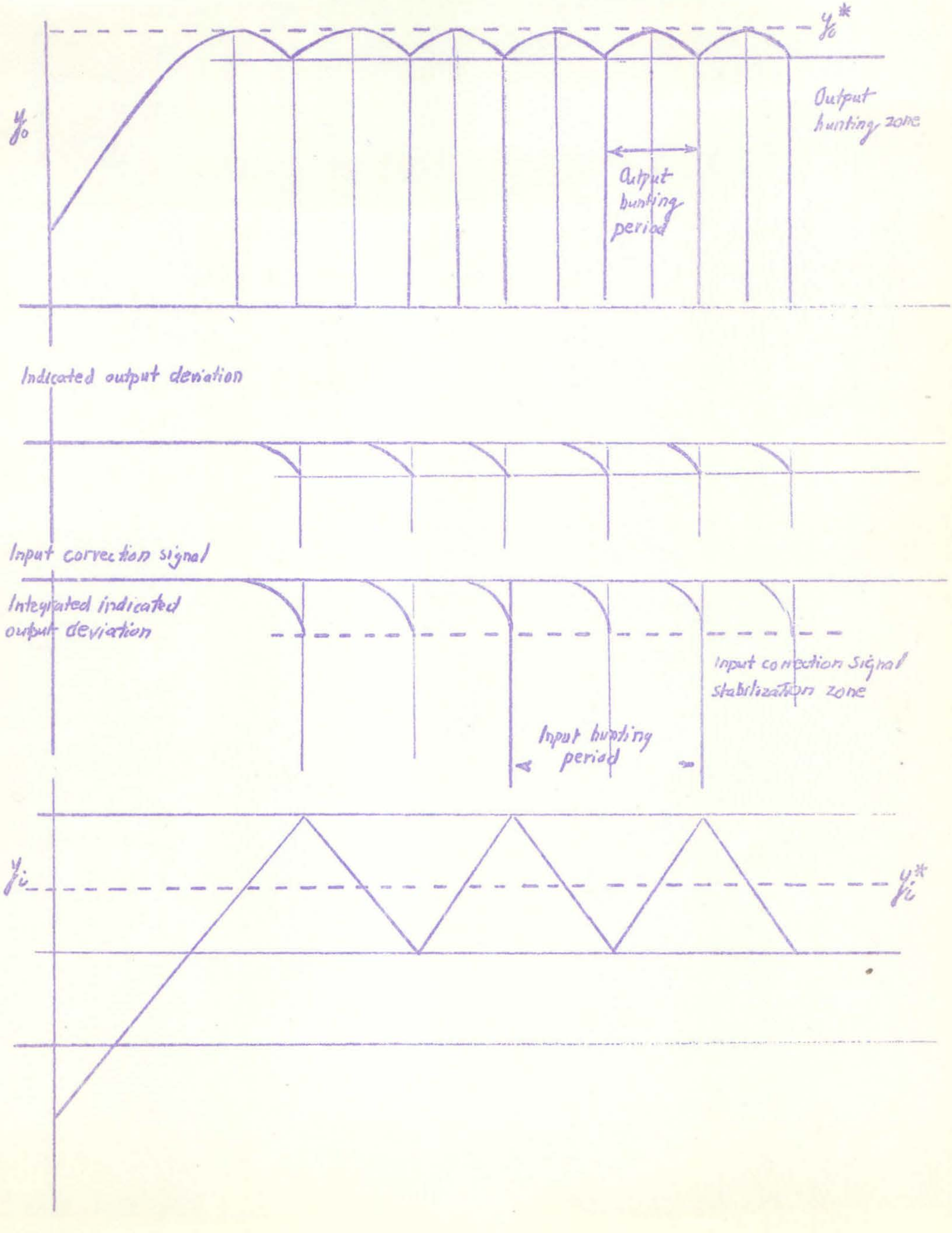

Controlled System
Controlled Input
Controlled Input Adjustor
Input Flow Rate
Input Converter
Input Converter Output
Output Coupler
Output
Controlled Output Setting
Optimalizing Controller
Input Adjustor Drive Drive Signal Control System
Input Correction Signal Generating System
Controlled Input Adjustor Drive
Controlled Input Adjustor Drive Signal
Controlled Input Adjustor Drive Drive Signal Generating System
Input Correction Signal
Input Correction Signal Generator
Smoothed Output Signal
Output Signal Smoothing System
Output Signal
Output Receiver
Indicated Optimum Output
Indicated Optimum Output Signal Generator
Stabilization Zone Setting
Indicated Deviation
Output Resetting Signal
Signal Resetting System
Indicated Optimum Output Resetting Signal

signal generating system. The indicated optimum output signal is supplied as one of the inputs to the input correction signal generator, which also receives the smoothed output signal from the output signal smoothing system. The difference between the output signal and the indicated optimum output signal is the indicated output deviation signal. The time integral of this deviation signal builds up until a stabilization zone limit is reached. When this limit is reached, the input correction signal is applied to the controlled input adjustor drive and reverses the sense of the controlled input rate. When this reversal occurs, the input adjustor drive reversal signal acts on the signal resulting system to reduce the signal representing the integral of the output deviation to zero. The resetting signals are supplied to the indicated optimum output signal generating system, and the input correction signal generator. The resultant effect of the action described is that the output of the controlled system remains within a hunting zone corresponding to the output signal hunting zone.

Peak holding type controllers operate without the use of a certain specified control frequency which is required in a continuous test signal controller and an output sampling controller. For this reason the peak holding type controller is more suitable than the other two types of controller for the control of operating systems with strong interferences.

Dynamic Characteristics of Optimalizing Systems

Since optimalizing controllers are designed to cope with the drift of the behavior of the controlled system, the variations of the input for this purpose must of necessity be an order faster than the drift variations of the input-output relation. For this reason, the dynamic analysis of the optimalizing controllers can be made under the assumption that such input-output relation is not influenced by the drift effect. We shall further assume that the input-output relation is also


y_o^*
y_o
Output hunting zone
Output hunting period
Indicated output deviation
Input correction signal
Integrated indicated output deviation
Input correction signal stabilization zone
Input hunting period
y_i
y_i^*

independent of the time rate of change of the input. In other words, the input-output relation to be used is quasi-steady. For internal combustion engines, this is fully justified because of the production of mechanical torque at constant speed is a gas dynamic and combustion process. The characteristic time for such process is very short indeed. Since the input-output relation is the only essentially non-linear element of the controlled system, the assumption of invariant input-output relation greatly simplifies the analysis of dynamic effects in optimalizing controllers.

Consider the continuous test function controllers first. The output y_0 (t) under a sinusoidal test function of Eq. (11.5) is given by Eq. (11.6). The net result of the combined effects of the band-pass filter and the phase adjuster is to remove the quasi-steady part of the output and the second harmonic. If λ_1 is the sensitivity of the system, then the output y_1 of the phase adjustor is

$$y_1 = -2Ka\lambda_1(y_i^0 - y_i^*)\sin\omega t \tag{11.10}$$

If λ_2 is the sensitivity of the rectifying multiplier, then the output y_2 of it is

$$y_2 = -2Ka^2\lambda_1\lambda_2\frac{\pi}{\omega}(y_i^0 - y_i^*) \tag{11.11}$$

If the rate of variation of the input is made to be proportional to y_2 with a factor of proportionality α, then

$$\frac{d}{dt}(y_i^0 - y_i^*) = -2Ka^2\alpha\lambda_1\lambda_2\frac{\pi}{\omega}(y_i^0 - y_i^*)$$

The characteristic time τ for the exponential approach to the optimum point is then

$$\tau = \frac{\omega}{2a^2\lambda_1\lambda_2\pi K} \tag{11.12}$$

Therefore the characteristic time is inversely proportional to sharpness of the optimum point in the input-output relation.

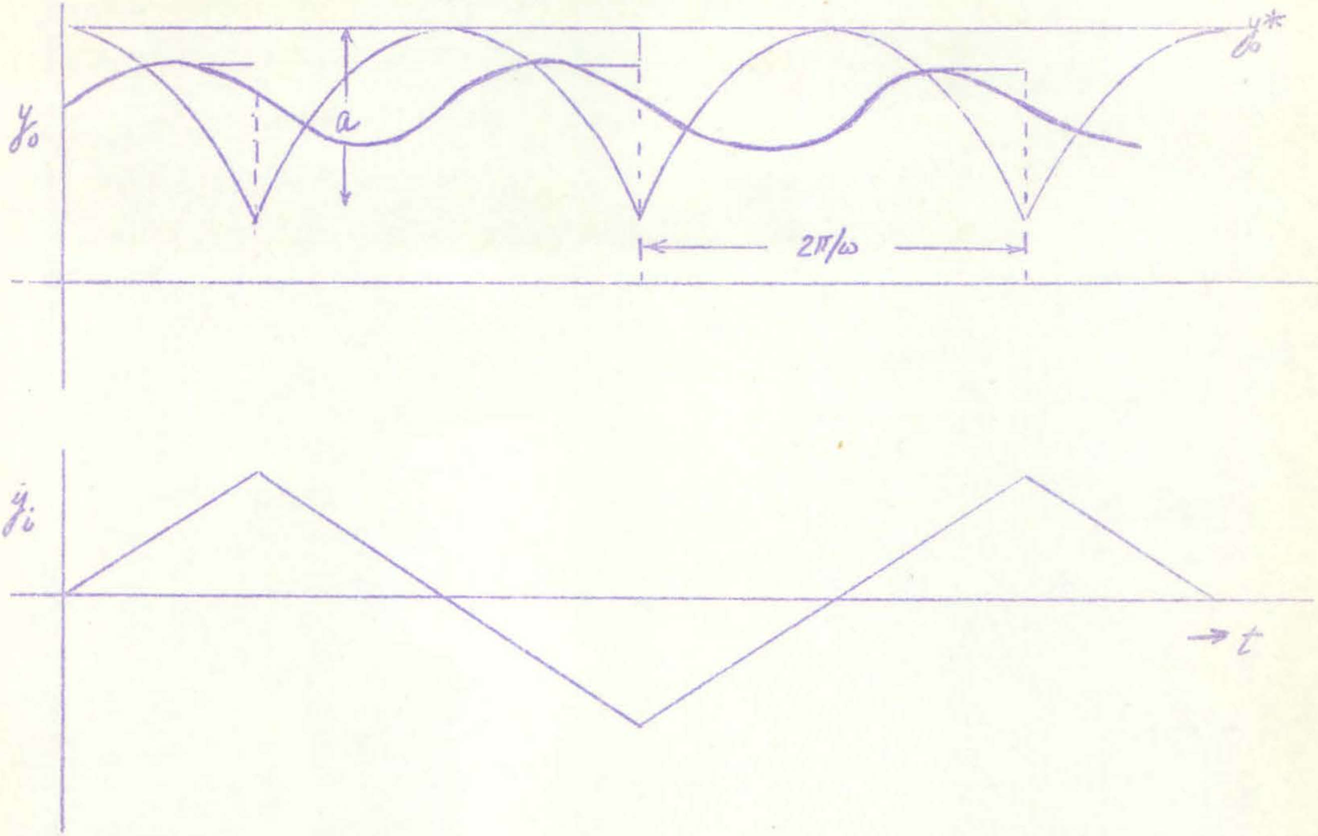
For optimalizing controllers of the output hunting type, the dynamic behavior is more difficult to analyse. However it is convenient to discuss system behavior in terms of groups of performance describing functions. These are divided into linear function groups, which can be expressed in terms of transfer functions, and non-linear function groups, which are not expressible as transfer functions. As far as control is concerned, the linear functions describe signal relating actions, while the essential optimalizing performance is associated with relationships established by the non-linear components. For the purposes of analysis it is convenient to form groups from individual functions without regard to whether or not the corresponding components are located in the controlled system or the controller. For a optimalizing controller then, the linear components may be considered as forming two groups:

a. The Input Linear Group, which consists of the controlled input adjustor drive signal generating system, the controlled input adjustor drive andtthe controlled input adjustor.

b. The Output Linear Group, which consists of the output coupler, the output receiver,and the output signal smoothing system.

Effect of a Response Delay in the Output Linear Group.

The following figure illustrates the action of an optimalizing control system in which the performance of the output linear system mby be represented in terms of a single first-order delay. The input variation is shown as a series of saw teeth corresponding to a constant input rate with reversals at evenly-spaced points. When no response delay exists in the input linear group and the controlled system characteristic is assumed to have a parabolic shape, the input converter



output correspond to the triangular wave input variation is the series of parabolic segments. Let the height of the segment or the output hunting zone be a, and the period be $2\pi/$, then the Fourier expansion of the variable output is

$$y_0 - y_0^* = a\left[\frac{1}{3} + \frac{4}{\pi^2}\cos\omega t - \frac{4}{4\pi^2}\cos 2\omega t + \frac{4}{9\pi^2}\cos 3\omega t \text{ ----}\right] \qquad (11.13)$$

Now if the transfer function of the output linear group is simply $\lambda_0/1+\tau_0 p$, the steady-state output X_0 from the output signal smoothing system is

$$y_0 = \frac{a}{\lambda_0}\left[-\frac{1}{3} + \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}\left\{\frac{e^{in\omega t}}{1+in\tau_0\omega} + \frac{e^{-in\omega t}}{}\right\}\right]$$

But

$$\frac{1}{1+in\tau_0\omega} = \frac{1-in\tau_0\omega}{1+n^2\tau_0^2\omega^2} = \frac{}{\sqrt{1+n^2\tau_0^2\omega_0^2}}\, e^{-i\tan^{-1} n\tau_0\omega}$$

Therefore

$$x_0 = \frac{a}{\lambda_0}\left[-\frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \frac{1}{\sqrt{1+n^2\tau_0^2\omega^2}} \cos\left(n\omega t - \tan^{-1} n\tau_0\omega\right)\right] \qquad (11.14)$$

For points corresponding to the peak of the parabolic areas, the Fourier series is rapidly convergent and only a few terms up to $n = m$ are necessary. Then if the delay time is short in comparison to the hunting period, say 1/6 of the period, then

$$\frac{1}{\sqrt{1+n^2\tau_0^2\omega^2}} \sim 1$$

$$\tan^{-1} n\tau_0\omega \sim n\tau_0\omega$$

Therefore for points near the peak of the curves,

$$x_0 \sim \frac{a}{\lambda_0}\left[-\frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{m} \frac{(-1)^{n+1}}{n^2} \cos n\omega(t-\tau_0)\right] \qquad (11.15)$$

If we designate the time interval between the optimum point of the y_0 and that of x_0 as the output linear group response delay. Then Eq. (11.15) shows that the output linear group response delay is equal to the characteristic time τ_0 of that group if the output linear group is a first order system. For several first order systems in series, the delay will be equal to the sum of the characteristic time in each system.

Let us consider the peak holding optimalizing controller. The time interval between the point of optimum y_o^* and the input drive reversal signal, i. e., half of the output hunting period T, is equal to the sum of output linear group response delay and the interval between the maximum of x_o and the input drive reversal signal. But the interval between the maximum of x_o and the input reversal signal is equal to one half of the output hunting period T_o when the output linear group delay is absent. Hence

$$T = T_o + 2\tau_o \tag{11.16}$$

Therefore the effect of the output linear group delay is to increase the hunting period by twice the characteristic time.

Since the input is driven at constant speed, the amplitude of input hunting is directly proportional to the period of hunting. The output hunting zone Δ, due to the parabolic input-output relation, is thus proportional to the square of the hunting period. Thus

$$\Delta = \Delta_o \left(\frac{T_o + 2\tau_o}{T_o} \right)^2 \tag{11.17}$$

where Δ_o is the output hunting zone with no response delay of the output linear group.

If v is the constant input rate, then $vT_o/2$ is the input deviation from y_i^*. Therefore using the parabolic input-output relation, we have

$$\Delta_o = K \left(vT_o/2 \right)^2 \tag{11.18}$$

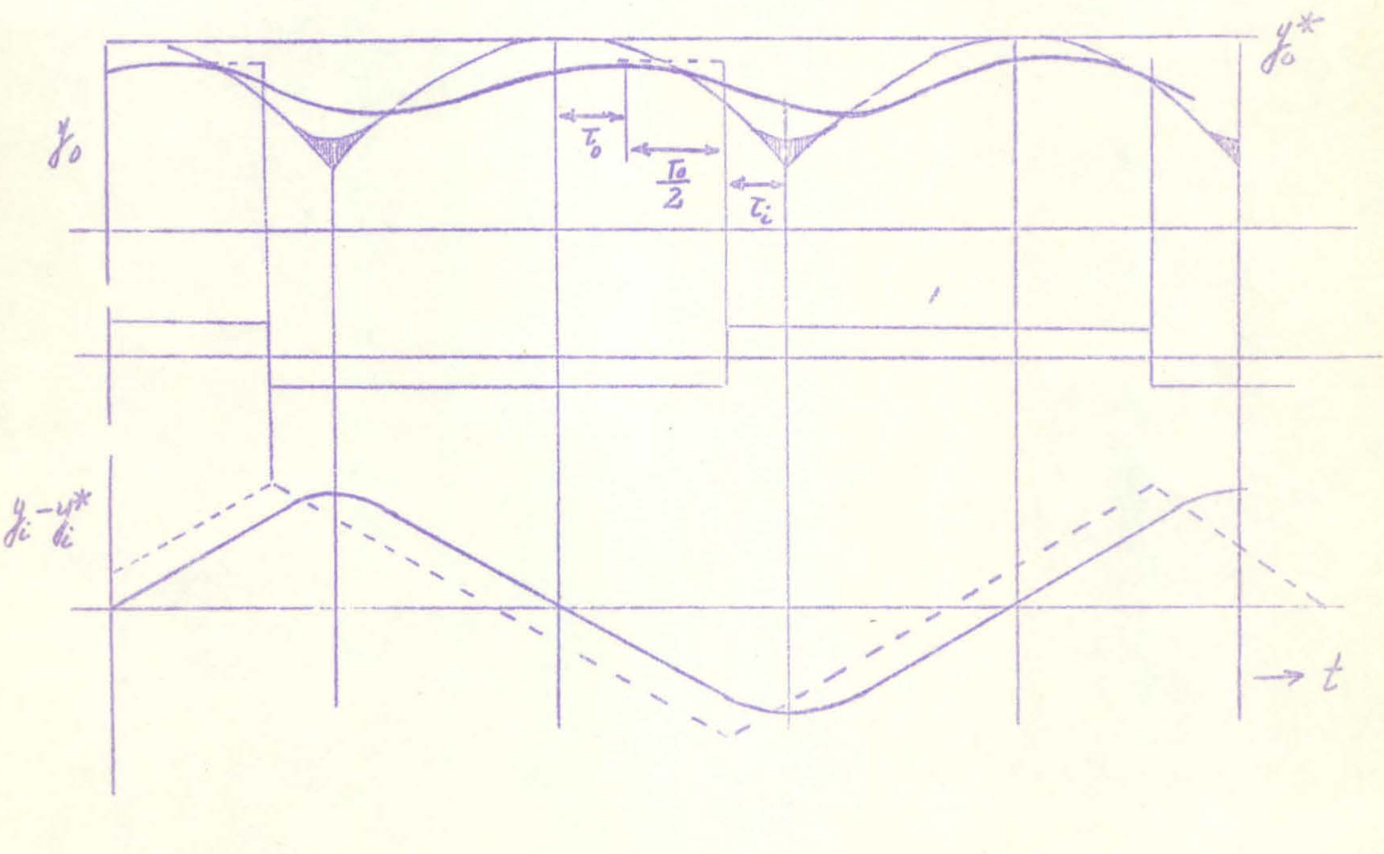
The output hunting loss D is thus

$$D = \frac{1}{3}\Delta = \frac{K v^2}{12}(T_0 + 2\tau_0)^2 \qquad (11.19)$$

With chosen T_0, Eq. (11.19) determines the drive speed ν for the input for specified output hunting loss D.

Combined Output Linear Group and Input Linear Group Response Delays

In general, any particular optimalizing system will have response delays in both the input linear group and the output linear group. When there is response delay in the input linear group, the actual input variation with time will not be the saw teeth curve with the



maxima and the minima at the instants of the input drive reversal. The actual input variation will have the sharp peaks and dips removed. By and by an entirely similar argument as given for the previous section, the main effect is an input response delay approximately equal to the characteristic time τ_i of the input linear group. Then it is easily seen that the output hunting period T is

$$T = T_0 + 2(\tau_0 + \tau_i) \tag{11.20}$$

Due to the rounding-off of the dips in the output curve, the actual hunting zone is less than the following quantity

$$\Delta_0 \left[T_0 + 2(\tau_0 + \tau_i)\right]^2 / T_0^2$$

Indicated Optimum Output Signal Stabilization Margin

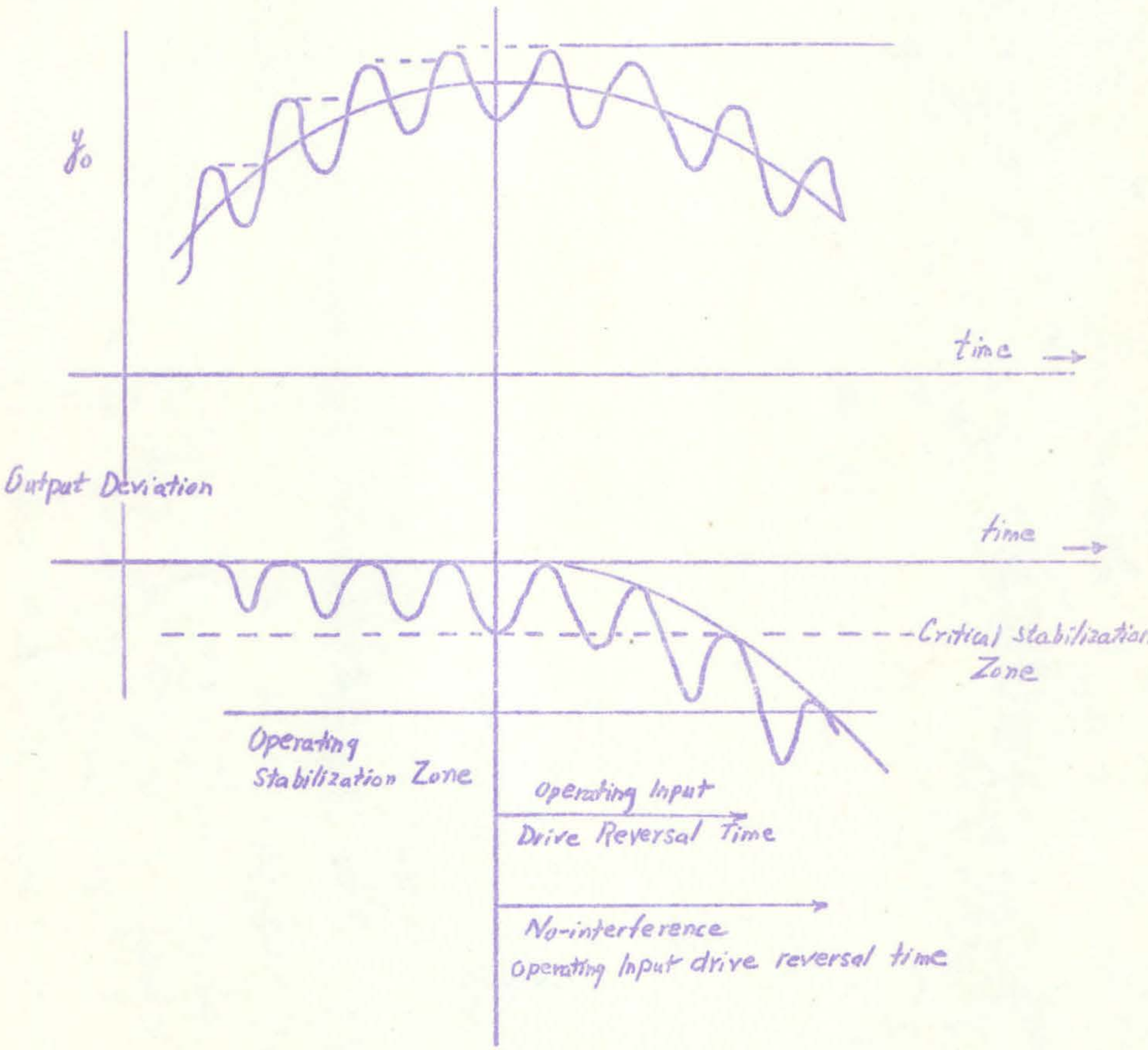
For peak holding optimalizing controllers, the indicated output deviation signal instantaneously returns to zero when the resetting action occurs. If no response delay effects are present in the controller or the controlled system, the indicated optimum output signal remains to the indicated output signal and the indicated output deviation signal holds to zero until the output signal passes through its maximum and starts to decrease. When response delays are present in the system, the input drive signal reversal occurs when the input correction signal exceeds the stabilization zone limit, just as this would happen with no response delays in the linear group. However, with response delays in the system the indicated output deviation signal would not hold to zero after the input drive reversal has occured. The reason for this effect is that, if the indicated optimum output signal is reset to equal to the smoothed output signal at the instant of input drive reversal, the indicated output deviation signal will continue to decrease toward the stabilization zone limit even though the controlled input is actually being adjusted toward the optimum point. In practice this action always has a de-stabilizing effect because the indicated deviation builds up even though the input is being driven in the proper direction to reach an optimum point. Satisfactory optimalizing control system performance is possible only if the build-up of this destablizing signal is kept within tolerable levels and prevented from reaching the stabilization zone limit.

One method is to introduce an <u>indicated optimum output signal stabilizing margin</u> in resetting the indicated optimum output signal. When this margin is used, the indicated optimum signal is not made just equal to the indicated output signal, but is reduced to a point lower than the indicated output signal by the stabilization margin. The dynamic characteristics of the resetting system are chosen so that the indicated optimum output signal is caused to approach the indicated output. This means that the use of a proper stabilizing margin coupled with suitable dynamics in the input correction signal generating system may be used to prevent a false input correction signal from being built up before the indicated correction signal has actually passed through the optimum level.

<u>Output Hunting Zone of a Peak Holding Type Optimalizing Controller</u>

For smooth output signals, there is no reason why the output hunting zone cannot be reduced to very small value so as to minimize the output hunting loss. In general, a stabilization zone is used primarily for the prevention of some undesirable effects caused by interferences. An example to show the relation between interference and stabilization is illustrated in the following figure. In this figure the upper plot shows the system output, which includes a no-interference output and an interference component of periodic nature with a fundamental frequency considerably higher than the hunting frequency which gives the shape of the no-interference output curve.

The lower plot shows the output deviation signal which is used as the input correction signal for the example represented by this figure.



If any of these ripples forming output deviation signals is allowed to exceed a stabilization zone setting, then an input rate reversal would occur because of the property of the peak holding type controller. In this situation the output would start to decrease before the actual optimum level is reached. To prevent this undesirable performance from happening, it is necessary to set the stabilization zone lower than a critical stabilization zone, which is defined as the stabilization zone barely sufficient to prevent an input rate reversal due to interference effects alone. An operating stabilization zone is the zone actually used in operation. The ratio between this stabilization zone and the critical stabilization zone is called the stabilization zone operating ratio.

In general, the output hunting zone of a system is of the same order of magnitude as the interferences of the output signal used to generate the indicated optimum output level. Therefore in order to reduce the output hunting zone it is necessary to smooth out the interference by the use of a properly designed filter. Another method of reducing the interference effects is to use the time integral of the output deviation signal as the input correction signal.

12. Linear Systems under Random Disturbances

We shall introduce another new element into our study of control in this chapter: The forcing function or the input will be considered, not as a definite specified function of time, but a random function which is specified only by certain statistical properties. Examples of such random input are the interference or "noise" spoken in the last chapter, and the aerodynamic response of an airplane wing in a turbulent atmosphere. The problem of noise is a problem of much research in connection with the problem of communication. There the central question is how to devise a system of code so that the inevitable noise effect can be minimized and the useful information is not lost. The problem of this chapter is somewhat different: The noise effect is the main output of the system. The aim is to design the system by appropriate servo circuit so as to achieve the desired characteristics of the output.

Statistical Description of Noise

Let us consider a device which generates a random function $y_1(t)$. Now to formulate the concept of a statistical description of such random function, we have to consider a great number of devices macrospically identical to the first. Such a group of devices is called an assembly. The random functions generated are $y_1(t)$, $y_2(t)$, $y_3(t)$, $\cdots\cdots$. At a definite time instant t, the y's are generally not equal. But we can ask for what fraction of the total number of cases y occurs in a given range y and $y + \Delta y$. This fraction will depend on y and t and will be proportional to Δy when Δy is small. It is written as $W_1(y,t)dy$ and $W_1(y,t)$ is the first probability function. Next we can consider all the pairs of values of y occuring at two given times t_1 and t_2. The fraction of the total number of pairs in which y occurs in the range $(y_1, y_1 + \Delta y_1)$ at t_1 and in the range $(y_2, y_2 + \Delta y_2)$ at t_2 is written as $W_2(y_1, t_1; y_2, t_2)\,dy_1\,dy_2$. $W_2(y_1, t_1; y_2, t_2)$ is called the second probability distribution.

It may be objected that observations of the noise on an assembly of devices can never be made, simply because of practical difficulty. Such observations are not necessary, however, when the noise output is stationary. This means that the statistical character of the noise is not time varying. If one observation is made of the noise for a very long time, all the information desired will be received. The record can be cut in pieces of length Θ (where Θ is long compared with all "periods" of the noise) and the different pieces can then be considered at the different records of an assembly of observations from which the different probability distributions can be determined. Furthermore, these distributions now become somewhat simpler. W_1 will be independent of t_1 and W_2 will be dependent only upon the difference $t = t_2 - t_1$. Thus for stationary random functions, we have the following descriptions of the stistical character:

$W_1(y)dy$ = probability of finding y between y and y + dy.

$W_2(y_1, y_2, t)dy_1\, dy_2$ = probability of finding a pair of values of y in the ranges $(y_1, y_1 + dy_1)$ and $(y_2, y_2 + dy_2)$ which are a time interval t apart from each other.

and higher probability distributions W_n.

It should be emphasized that these probability distributions represent everything that can be found out about the random function, and one may therefore say that the random function is defined by these distributions. Of course, the functions W_n are not arbitrary and unrelated to each other; they must fulfill the following three obvious conditions:

1) $W_n \geqq 0$, because the W_n are probability densities,

2) $W_n(y_1, t_1; y_2, t_2; \cdots; y_n, t_n)$ must be a symmetric function in the set of variables $y_1, t_1; y_2, t_2; \cdots; y_n, t_n$, since W_n is a joint probability, and

3)

$$W_k(y_1, t_1; y_2, t_2; \cdots; y_k, t_k)$$

$$= \int \cdots \int dy_{k+1}\, dy_{k+2} \cdots dy_n \cdot W_n(y_1, t_1; y_2, t_2; \cdots; y_k, t_k; y_{k+1}, t_{k+1}; \cdots; y_n, t_n) \; . \tag{12.1}$$

Since each function W_n must imply all the previous W_k, with $k < n$. It should be noted that in Eq. (12.1), the integration over all possible values of y_{k+1} also eliminate the variable t_{k+1} in W_n. This is a feature of probability distributions. For instance

$$\int W_1(y_1, t_1)\, dy_1 = 1 \tag{12.2}$$

due to the definition of probability.

Average Values

From the first probability distribution $W_1(y,t)$ the average value of y can be found

$$\bar{y}(t) = \int y W_1(y,t)\, dy \tag{12.3}$$

Clearly, this average value will in general depend on the time t. It can be determined by averaging, at time t, the noise $y_1(t)$, $y_2(t)$ .... of the assembly. It will be spoken of, therefore, as the assembly average, and indicated by a bar. It must be distinguished from the time average, defined and denoted by

$$\tilde{y} \equiv \lim_{\Theta \to \infty} \frac{1}{\Theta} \int_{-\Theta/2}^{\Theta/2} y(t)\, dt \tag{12.4}$$

This will be independent of the time, of course, but will, in general, differ for the various functions $y_1(t)$, $y_2(t)$, ... of the assembly. We can still average over the assembly; then the same result will be obtained as the time average of $\bar{y}$. Or

$$\tilde{\bar{y}} = \bar{\tilde{y}} \tag{12.5}$$

Only for a stationary process will the two ways of averaging give the same result, since then $\bar{y}$ will be independent of t and $\tilde{y}$ will be the same for all members of the assembly.

The same distinction must, of course, be made for the average values of functions of y . Usually it will be clear from the context which kind of averaging is meant. Especially important are the different moments of the distribution W_1 , defined by

$$m_n = \bar{y}^n = \int y^n W_1(y,t)\,dy \tag{12.6}$$

From the first and the second moments there is derived the fluctuation, variance, or the mean deviation

$$\begin{aligned}\overline{(y-\bar{y})^2} &= \int (y-\bar{y})^2 W_1(y,t)\,dy \\ &= \int (y^2 - 2\bar{y}y + \bar{y}^2) W_1(y,t)\,dy = \overline{y^2} - (\bar{y})^2\end{aligned} \tag{12.7}$$

which is a measure of the width of the distribution $W_1(y)$ about the average value $\bar{y}$. From the third moment an idea of the skewness of the probability distribution can be made. More and more information about $W_1(y)$ can be deduced when additional moments are known. The problem whether or not the knowledge of all moments determines the probability distribution uniquely is a famous one, but it will not concern us here. In certain instances this is actually the case, as, for example, when $m's$ fulfill the relations

$$\left.\begin{aligned} m_{2k+1} &= 0 \\ m_{2k} &= 1\cdot 3\cdot 5\cdots(2k-1)(m_2)^k \end{aligned}\right\} \tag{12.8}$$

Then $W_1(y)$ is the Gaursian distribution

$$W_1(y) = \frac{1}{\sqrt{2\pi m_2}}\, e^{-\frac{y^2}{2m_2}} \tag{12.9}$$

Of special importance is the combination of the moments embodied in the so-called <u>characteristics function</u>

$$\phi_1(s) = \overline{e^{isy}} = \sum_{n=0}^{\infty} \frac{(is)^n}{n!} m_n = \int e^{isy} W_1(y)\,dy \tag{12.10}$$

The significance of the characteristic function lies especially in the following two theorems:

1. The characteristic function determines uniquely the probability distribution. In fact, from the Fourier integral theorem it follows that

$$W_1(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isy} \phi_1(s)\,ds \tag{12.11}$$

2. If the characteristic functions of two independent random variables y and z are $\phi(s)$ and $\psi(s)$, then the characteristic function of the distribution of the sum y + z is given by the product $\phi(s)\psi(s)$. This can be seen as follows: The probability of the simultaneous occurance of independent events is the product of individual probabilities. Thus

$$W_1(y+z) = W_1(y)W_1(z)$$

Then the moments of $W_1(y+z)$ is

$$\iint (y+z)^n W_1(y+z)\,dy\,dz = \iint (y+z)^n W_1(y)W_1(z)\,dy\,dz$$

Thus the characteristic function $\chi(s)$ for the distribution of y+z is

$$\chi(s) = \iint e^{is(y+z)} W_1(y)W_1(z)\,dy\,dz = \phi(s)\psi(s) \tag{12.12}$$

Let us now turn to the second distribtuion function $W_2(y_1, t_1; y_2, t_2)$ The most important average value derived from it is

$$\overline{y_1 y_2} = \iint y_1 y_2\, W_2(y_1, t_1; y_2, t_2)\,dy_1\,dy_2 \tag{12.13}$$

In general this will be a function of t_1 and t_2. Letting $t_2 = t_1 + \tau$, we can perform an additional time average over t_1, and then obtain a function of τ

$$R(\tau) = \lim_{\Theta \to \infty} \frac{1}{\Theta} \int_{-\Theta/2}^{\Theta/2} \overline{y_1 y_2}\,(t_1, t_1 + \tau)\,dt_1 = \widetilde{\overline{y_1 y_2}} \tag{12.14}$$

The same function R(t) is obtained, of course, by taking the assembly average of

$$\widetilde{y(t_1)\,y(t_1+\tau)} = \lim_{\Theta \to \infty} \frac{1}{\Theta} \int_{-\Theta/2}^{\Theta/2} y(t_1)\,y(t_1+\tau)\,dt_1 \tag{12.15}$$

For a stationary process Eqs. (11.13) and (11.15) give the same result. The function $R(\tau)$ gives a measure for the correlation between successive values of y and is therefore called the correlation function. When $y(t_1)$ and $y(t_2)$ are independent of each other

$$W_2(y_1, t_1 ; y_2, t_2) = W_1(y_1, t_1)\,W_1(y_2, t_2)$$

and

$$\overline{y_1 y_2} = \overline{y}_1 \cdot \overline{y}_2$$

It is often that this situation will occur when the time interval $z = t_2 - t_1$ is sufficiently large. For $\tau = 0$, it is obvious that

$$\left(\overline{y_1 y_2}\right)_{\tau=0} = \overline{y^2} = \int y^2\,W_1(y, t)\,dy \tag{12.16}$$

It is sometimes convenient to work with the function

$$\rho(\tau) = \frac{\overline{(y_1-\bar{y})(y_2-\bar{y})}}{\overline{(y-\bar{y})^2}} \qquad (12.17)$$

which is called the normalized correlation function. It is seen that $\rho(0) = 1$, and $\rho(t) \leqq \rho(0)$. As $t \to \infty, \rho(t) \to 0$.

<u>Power Spectrum</u>

Of special significance for our applications is the notion of the spectrum of a random function. Let us suppose that a function y(t) is observed for a long time Θ . Assuming that y(t) = 0 outsie the time interval Θ , then y(t) can be developed in a Fourier integral

$$y(t) = \int_{-\infty}^{\infty} A(\omega)\, e^{i\omega t}\, d\omega \qquad (12.18)$$

In fact the amplitude function $A(\omega)$ can be calculated by the inversion. theorem as

$$A(\omega) = \frac{1}{2\pi}\int_{-\Theta/2}^{\Theta/2} y(t)\, e^{-i\omega t}\, dt \qquad (12.19)$$

If we denote ( )* as the complex conjugate of ( ), then since is real,

$$A^*(\omega) = A(-\omega) \qquad (12.20)$$

Now we can calculate the time average $\widetilde{y^2}$ as

$$\widetilde{y^2} = \lim_{\Theta\to\infty}\frac{1}{\Theta}\int_{-\Theta/2}^{\Theta/2} y^2(t)\,dt = \lim_{\Theta\to\infty}\frac{1}{\Theta}\int_{-\Theta/2}^{\Theta/2} dt \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega\, d\omega'\, A(\omega)A(\omega')\,e^{i(\omega+\omega')t}$$

By the substitution

$$\omega'' = -\omega'$$

$$\widetilde{y^2} = \lim_{\Theta\to\infty}\frac{1}{\Theta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega\, d\omega''\, A(\omega)A^*(\omega'')\int_{-\Theta/2}^{\Theta/2} e^{i(\omega-\omega'')t}\,dt$$

$$= \lim_{\Theta\to\infty}\frac{2}{\Theta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\omega)A^*(\omega'')\,\frac{\sin(\omega-\omega'')\Theta/2}{\omega-\omega''}\,d\omega\, d\omega''$$

If we now substitute

$$\xi = \frac{\Theta}{2}(\omega-\omega'')$$

then

$$\omega'' = \omega - \frac{2\xi}{\Theta}$$

Consequently

$$\widetilde{y^2} = \lim_{\Theta\to\infty}\frac{2}{\Theta}\int_{-\infty}^{\infty} A(\omega)\,d\omega\int_{-\infty}^{\infty} A^*\left(\omega-\frac{2\xi}{\Theta}\right)\frac{\sin\xi}{\xi}\,d\xi$$

$$= \left[\lim_{\Theta\to\infty}\frac{1}{\Theta}\int_0^{\infty}|A(\omega)|^2 d\omega\right]4\int_{-\infty}^{\infty}\frac{\sin\xi}{\xi}\,d\xi$$

Therefore (12.21)

$$\widetilde{y^2} = \int_0^{\infty} f(\omega)\,d\omega$$

where

$$f(\omega) = \lim_{\Theta \to \infty} \frac{4\pi}{\Theta} |A(\omega)|^2 \tag{12.22}$$

The function $f(\omega)$ is called the power spectrum of the random function. This relation between the power spectrum and the Fourier coefficient is called the Parseval theorem.

Let us consider next the average value

$$\widetilde{y(t)\, y(t+\tau)} = \lim_{\Theta \to \infty} \frac{1}{\Theta} \int_{-\Theta/2}^{\Theta/2} y(t) y(t+\tau) dt$$

By substituting the Eq. (11.18).

$$\widetilde{y(t)y(t+\tau)} = \lim_{\Theta \to \infty} \frac{1}{\Theta} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\omega) A(\omega') e^{i\omega\tau} d\omega\, d\omega' \int_{-\Theta/2}^{\Theta/2} e^{i(\omega+\omega')t} dt$$

Then, by a similar argument as before,

$$\widetilde{y(t)y(t+\tau)} = \int_0^{\infty} f(\omega) \cos \omega\tau \, d\omega \tag{12.23}$$

When $\tau = 0$, Eq. (11.23) reduces to Eq. (11.21). According to the inversion theorem,

$$f(\omega) = \frac{2}{\pi} \int_0^{\infty} \widetilde{y(t)\, y(t+\tau)} \cos \omega\tau \, d\tau \tag{12.24}$$

All this holds for any function $y(t)$. Let us assume, now, that we have a random process and that $y(t)$ is a member of the assembly of functions $y_1(t)$, $y_2(t)$, $\cdots$. Each of these functions can be developed in a Fourier integral, and the corresponding $f(\omega)$ can be averaged over the assembly. The resulting $\overline{f(\omega)}$ will be called the spectrum of the random process. Thus

$$R(\tau) = \int_0^\infty \overline{f(\omega)} \cos \omega\tau \, d\omega \tag{12.25}$$

$$\overline{f(\omega)} = \frac{2}{\pi} \int_0^\infty R(\tau) \cos \omega\tau \, d\tau \tag{12.26}$$

Eq. (11.21) can be also averaged over the assembly, then

$$\overline{\widetilde{y^2}} = \int_0^\infty \overline{f(\omega)} \, d\omega \tag{12.27}$$

For a stationary process the averaging over the assembly can be omitted, since each member $y_2(t)$ will lead to the same power spectrum $f(\omega)$.

The power spectrum $f(\omega)$ may contain singular peaks of the well-known Dirac δ-function type. This certainly occurs, for instance, when $\overline{y}$ is not zero or, in electrical engineering language, when there is a d.c. term. Then

$$f(\omega) = 2\tilde{y}^2 \delta(\omega) + f_1(\omega) \tag{12.28}$$

where $\delta(x)$ is the Dirac δ-function, i.e.,

$$\delta(-x) = \delta(x) = 0 \quad \text{for} \quad x \neq 0, \quad \delta(x) = \infty \quad \text{for} \quad x = 0$$

such that

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1, \qquad \int_{0}^{\infty} \delta(x)\,dx = \frac{1}{2}$$

For pure noise, the peak at $\omega = 0$, corresponding to the d.c. term will usually be the only peak, so that $f_1(\omega)$ will be a regular function, representing the really continuous spectrum. When there are noise and a signal, the power spectrum $f(\omega)$ will consist of a continuous spectrum and a number of peaks at the discrete frequencies ω_i of the signal.

Simple Forms of the Power Spectrum

For isotropic turbulence, it is known that the fundamental correction functions are $R_1(\tau)$ and $R_2(\tau)$ is the correction of the fluctuating velocity parallel to the mean flow direction at a time interval τ. $R_2(\tau)$ is the correction of the fluctuating velocity normal to the mean flow direction. These correction functions are approximated by

$$R_1(\tau) = R_1(0)\, e^{-\tau U/L} \tag{12.29}$$

$$R_2(\tau) = R_2(0)\, e^{-\tau U/L}\left(1 - \frac{1}{2}\frac{\tau U}{L}\right) \tag{12.30}$$

where U is the mean flow velocity and L is the scale of turbulence. From these equations, the corresponding power spectra $f_1(\omega)$ and $f_2(\omega)$ can be calculated by Eq. (12.26). They are

$$\left.\begin{aligned} f_1(\omega) &= f_1(0)\,\frac{1}{1+\xi^2} \qquad (12.31)\\ f_2(\omega) &= f_2(0)\,\frac{1+3\xi^2}{(1+\xi^2)^2} \qquad (12.32)\end{aligned}\right\}\ \xi = \frac{\omega L}{U}$$

If the correctin function $R(\tau)$ is a Gaussian curve,

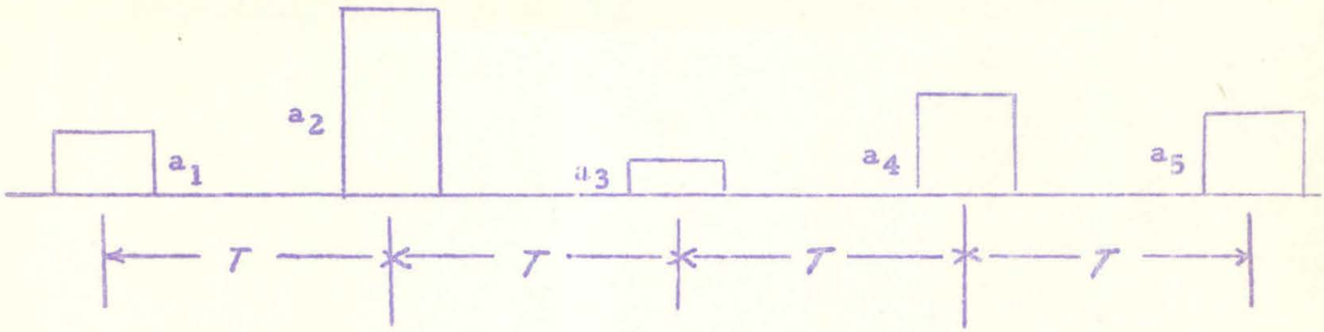
$$R(\tau) = R(0)\, e^{-\alpha^2\tau^2} \tag{12.33}$$

then the power spectrum $f(\omega)$ is

$$f(\omega) = f(0)\, e^{-\frac{\omega^2}{4\alpha^2}} \tag{12.34}$$

When $\alpha \rightarrow \infty$, the correlation function becomes a δ-function, i.e., the random process is uncorrelated. Then $f(\omega) = f(0) =$ constant, i.e, the power spectrum is a constant. This type of random process is called white noise and often describes the naturally generated random variations.

It is not necessary, of course, to calculate the power spectrum by means of the correlation function. Sometimes it is just as easy to derive the spectrum directly. Let us consider, for example, the case where y(t) consists of a series of pulses that have identical shape and a constant repetition frequency but whose heights vary according to some probability



distribution. If F(t) represents a single pulse of unit height, then

$$y(t) = \sum_k{}' a_k F(t - kT) \tag{12.35}$$

where T is the spacing of the pulses. Then according to Eq. (12.19),

$$A(\omega) = \frac{1}{2\pi}\int_{-NT}^{NT} y(t)\, e^{-i\omega t}\, dt = \frac{1}{2\pi}\int_{-NT}^{NT} \sum_k{}' a_k F(t-kT)\, e^{-i\omega t}\, dt$$

$$= \sum_{-N}^{N}{}' a_k \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t-kT)\, e^{-i\omega t}\, dt$$

$$= \sum_{-N}^{N}{}' a_k \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\xi)\, e^{-i\omega\xi - i\omega kT}\, d\xi = B(\omega)\sum_{-N}^{N}{}' a_k e^{-i\omega kT} \tag{12.36}$$

where

$$B(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\xi)\, e^{-i\omega\xi}\, d\xi \tag{12.37}$$

It has been assumed that there are approximately (2N + 1) pulses between the times -NT to +NT. Then the power spectrum $f(\omega)$ is, according to Eq. (12.22)

$$f(\omega) = 4\pi \lim_{N\to\infty} \frac{1}{2NT} |A(\omega)|^2$$

$$= \frac{4\pi}{T} |B(\omega)|^2 \lim_{N\to\infty} \frac{1}{2N} \left\{ \sum_{-N}^{N} \sum_{-N}^{N} a_k a_{k'} e^{-i\omega(k-k')T} \right\} \tag{12.38}$$

Let the mean value of a_k be $\bar{a}$, the mean square be $\overline{a^2}$. Then

$$a_k = (a_k - \bar{a}) + \bar{a}$$

$$a_{k'} = (a_{k'} - \bar{a}) + \bar{a}$$

So

$$a_k a_{k'} = (a_k - \bar{a})(a_{k'} - \bar{a}) + \bar{a}\left[(a_k - \bar{a}) + (a_{k'} - \bar{a})\right] + \bar{a}^2$$

Now we take an assembly average of Eq. (12.38), then

$$\lim_{N\to\infty} \frac{1}{2N} \left\{ \sum_{-N}^{N} \sum_{-N}^{N} \overline{(a_k - \bar{a})(a_{k'} - \bar{a})}\; e^{-i\omega(k-k')T} \right\} = \overline{a^2} - (\bar{a})^2$$

$$\lim_{N\to\infty} \frac{1}{2N} \left\{ \sum_{-N}^{N} \sum_{-N}^{N} \bar{a}\,\overline{\left[(a_k - \bar{a}) + (a_{k'} - \bar{a})\right]}\; e^{-i\omega(k-k')T} = 0$$

and $$\lim_{N\to\infty}\frac{1}{2N}\left\{\sum_{-N}^{N}\sum_{-N}^{N}(\bar{a})^2 e^{-i\omega(k-k')T}\right\} = \lim_{N\to\infty}\frac{(\bar{a})^2}{2N+1}\left|\sum_{-N}^{N} e^{-i\omega kT}\right|^2$$

Therefore

$$\overline{f(\omega)} = \frac{4\pi}{T}\left|B(\omega)\right|^2\left\{\left[\overline{a^2}-(\bar{a})^2\right]+(\bar{a})^2\lim_{N\to\infty}\frac{1}{2N+1}\left|\sum_{-N}^{N} e^{-i\omega kT}\right|^2\right\} \qquad (12.39)$$

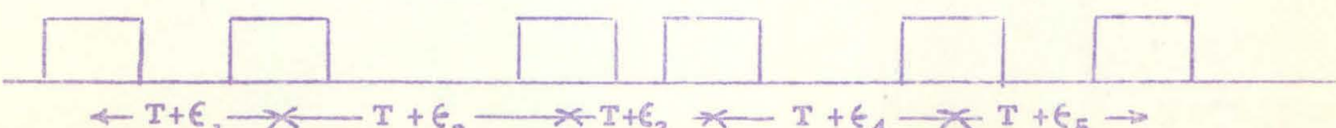
The sum inside the braces will be $2N + 1$ for $\omega = 2n\pi/T$ when n is an integer; hence in the limit $\overline{f(\omega)}$ will be infinite. For other values of ω the sum will be oscillatory, and for $N \to \infty$ the limiting value will be zero. Clearly, the limit has the character of a series of peaks, or δ - functions, at the frequency $2n\pi/T$, and $\overline{f(\omega)}$ can be written as

$$\overline{f(\omega)} = \frac{4\pi}{T}\left|B(\omega)\right|^2\left\{\left[\overline{a^2}-(\bar{a})^2\right]+\frac{(\bar{a})^2}{T}\sum_{n=0}^{\infty}\delta(\omega-n\omega_0)\right\} \qquad (12.40)$$

where $$\omega_0 = 2\pi/T$$

Therefore, a continuous spectrum is obtained that has the same shape as the power spectrum of a single pulse. The total intensity is determined by the fluctuation $\overline{a^2} - (\bar{a})^2$ of the pulse heights. There is, in addition, a discrete spectrum at the spectrum $n\omega_0$, where the intensities are also determined by the spectrum of the single pulse.

Let us consider next a series of pulses that have identical shape and height but a repetition period $T + \epsilon$ .varying around an average value T



according to some probability distribution $P(\epsilon)$. Now

$$y(t) = \sum_k' F(t - kT - \epsilon_k) \tag{12.41}$$

ϵ_k is the deviation of the kth spacing from T, so $\bar{\epsilon} = 0$. Let $\phi(\omega)$ be the characteristic function for ϵ, i.e.,

$$\phi(\omega) = \int_{-\infty}^{\infty} P(\epsilon)\, e^{i\omega\epsilon}\, d\epsilon \tag{12.42}$$

Then a similar calculation as before will give the power spectrum as

$$\overline{f(\omega)} = \frac{4\pi}{T} |B(\omega)|^2 \left\{ \left[1 - |\phi(\omega)|^2\right] + \frac{|\phi(\omega)|^2}{T} \sum_{n=0}^{\infty}{}' \delta(\omega - n\omega_0) \right\} \tag{12.43}$$

Here, the shape of the continuous spectrum and the intensitives of the discrete spectrum are no longer determined solely by the spectrum of the single pulse but depend also on the function $\phi(\omega)$.

Suppose that the function y(t) consists of pieces of the function $e^{i\omega_0 t}$, the length l_i of these pieces being distributed according to the probability distribution $P(l)$. Let us suppose further that at the end of each piece thepphase changes and that these phase changes are governed by the probability distribution $Q(\alpha)$. Then it can be shown* that the power spectrum is

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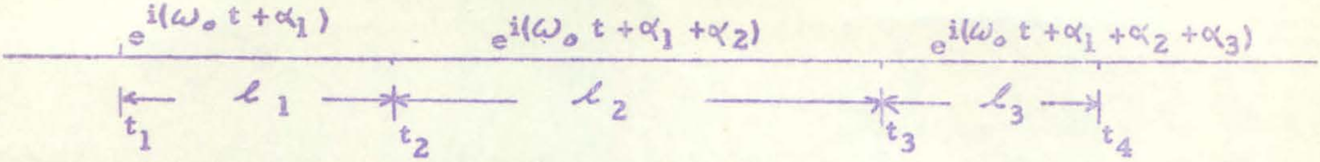
* Lawson V. Uhlenbeck, "Threshold Signals" McGraw Hill (1950) p. 45.



$$\overline{f(\omega)} = \frac{1}{2\pi}\left[\overline{y^2}-(\bar{y})^2\right] \frac{1-A+(A^2+B^2-1)\phi(\omega)-(A^2+B^2-A)\left[\phi^2(\omega)+\psi^2(\omega)\right]}{\frac{1}{4}\bar{l}\,(\omega-\omega_0)^2\left\{\left[1-A\phi(\omega)+B\psi(\omega)\right]^2+\left[A\psi(\omega)+B\phi(\omega)\right]^2\right\}} \tag{12.44}$$

$$A+iB = \int_{-\pi}^{\pi} d\alpha\, Q(\alpha)\, e^{i\alpha}$$

$$\phi(\omega)+i\psi(\omega) = \int_0^\infty P(l)\, e^{i(\omega_0-\omega)l} dl \tag{12.45}$$

and

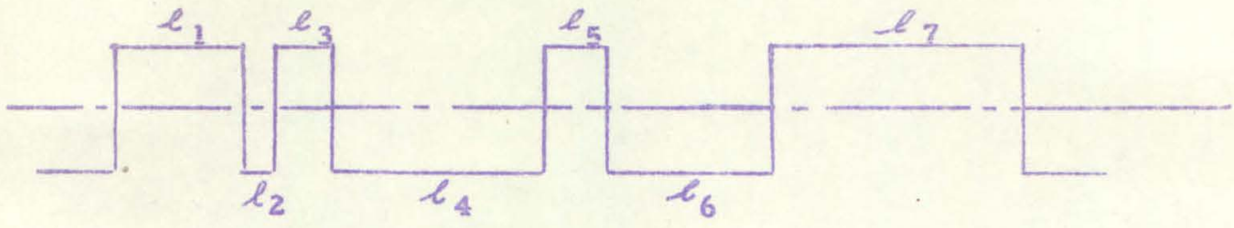
$$\bar{l} = \int_0^\infty P(l)\, l\, dl$$

Some special cases are of interest. Let $\omega_0 = 0$, and $Q(\alpha) = \delta(\alpha-\pi)$, so $A = -1$, $B = 0$; this leads to a step curve of height ± 1, in which the length of steps are distributed according to the probability law $P(l)$. Then from Eq. (12.44) we have, since $\overline{y^2} = 1$, $\bar{y} = 0$

$$f(\omega) = \frac{4}{\pi \bar{l}\,\omega^2}\,\frac{1-\left[\phi^2(\omega)+\psi^2(\omega)\right]}{\left[1+\phi(\omega)\right]^2+\psi^2(\omega)} \tag{12.46}$$

where now, of course,

$$\phi(\omega)+i\psi(\omega) = \int_0^\infty P(l)\, e^{-i\omega l} dl \tag{12.47}$$



If $P(\ell)$ has the Poisson's distribution

$$P(\ell) = \beta e^{-\beta \ell} \qquad \bar{\ell} = 1/\beta \tag{12.48}$$

then

$$\phi(\omega) + i\psi(\omega) = \beta \int_0^\infty e^{-(\beta + i\omega)\ell} d\ell = \frac{\beta^2 - i\beta\omega}{\beta^2 + \omega^2}$$

So the normalized power spectrum for this special case is

$$f(\omega) = \frac{1}{2\pi} \frac{8\beta}{4\beta^2 + \omega^2} \tag{12.49}$$

Probability of Large Deviations from the Mean

It will often become necessary to ask not only for the mean value $\bar{y}$ or $\overline{y^2}$, but also for the probability that a certain value of y is exceeded. This question is natural for strength considerations. If the probability distribution is known, then the answer is very simple:

$$P[|y| > k] = \int_{-\infty}^{-k} W_1(y)\, dy + \int_k^\infty W_1(y)\, dy \tag{12.50}$$

where P denote the probability of the event specified in the bracket.

It is, however, possible to give general but broad estimates of the probability. For instance if g(y) is a non-negative function of the random variable y_1 then since $W_1(y)$ is by definition non-negative,

$$\overline{g(y)} = \int_{-\infty}^{\infty} g(y) W_1(y)\, dy \geqq K \int_{\text{over } y \text{ when } g(y) \geqq K} W_1(y)\, dy$$

But the last integral is just $P\left[g(g) \geq K\right]$. Thus

$$P\left[g(y) \geqq K \right] \leqq \frac{\overline{g(y)}}{K} \tag{12.51}$$

This is the so-called Cheby shev's inequality. Now let

$$g(y) = (y - \bar{y})^2$$

Then

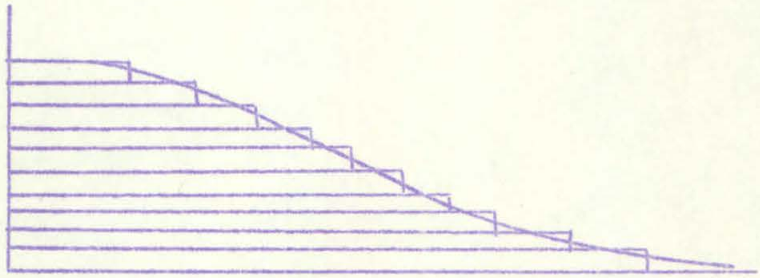
$$\overline{g(y)} = \sigma^2, \qquad \sigma + \bar{y} = \sqrt{\overline{y^2}} \tag{12.52}$$

where σ is the mean deviation from the mean. Let $K = k^2 \sigma^2$, then Eq. (12.36) gives the Bienaymé- Chebyshev in equality

$$P\left[|y - \bar{y}| \geqq k\sigma \right] \leqq \frac{1}{k^2} \tag{12.53}$$

The Bienaymé- Chebyshev inequality is known to be too broad for most practical applications, and the upper limit given is, in general, much too high. A sharper estimation can be given for $W_1(y)$

that has only a single maximum, the so-called mode. This estimate for "unimodal" distribution is due to Gauss (1821). To prove Gauss' inequality, let us consider a function $w(x)$ which is monotonically decreasing in the range $x \geqq 0$. It is seen that $w(x)$ can be considered as a superposition of functions which are constant from $x = 0$ to $x = x_0$ and zero for $x > x_0$. Let $v(x) = 1$ for $0 \leqq x \leqq x_0$ and $v(x) = 0$ for $x > x_0$.



Then for any $K > x_0$,

$$K^2 \int_K^\infty v(x)\,dx = 0$$

But if $0 < K \leqq x_0$,

$$K^2 \int_K^\infty v(x)\,dx = K^2(x_0 - K)$$

The maximum of this quantity for K within the range is $\frac{4}{27} x_0^3$. Thus it is generally true that

$$K^2 \int_K^\infty v(x)\,dx \leqq \frac{4}{9} \int_0^\infty x^2 v(x)\,dx$$

By superposition, we have then

$$K^2 \int_K^\infty w(x)dx \leqq \frac{4}{9} \int_0^\infty x^2 w(x) dx$$

Now consider any unimodal distribution with the abscissa $x = y - y_0$. y_0 is the mode, then

$$K^2 \int_K^\infty W_1(x)\,dx \leqq \frac{4}{9} \int_0^\infty x^2 W_1(x)\,dx$$

and

$$K^2 \int_{-\infty}^{-K} W_1(x)\,dx \leqq \frac{4}{9} \int_{-\infty}^{0} x^2 W_1(x)\,dx$$

By adding these expressions,

$$K^2 P[\,|y-y_0| \geqq K] \leqq \frac{4}{9} \nu^2$$

where ν is the mean deviation from the mode. Or

$$\nu^2 = \overline{(y-y_0)^2} = \int_{-\infty}^{\infty} (y-y_0)^2 W_1(y)\,dy \qquad (12.54)$$

Let $K = k\nu$, then we obtain the Gauss' inequality,

$$P\left[\,|y-y_0| \geqq k\nu\right] \leqq \frac{4}{9k^2} \qquad (12.55)$$

If the <u>distribution is symmetrical</u>, then $\nu = \sigma$, $y_0 = \bar{y}$, Eq. (12.55) reduces to

$$P[|y-\bar{y}| \geq k\sigma] \leq \frac{4}{9k^2} \tag{12.56}$$

Eq. (12.56) is much sharper than Eq. (12.51).

Often it will be possible to assume, at least approximately, a Gaussian distribution. Then by using the asymptotic expansion of the error function, it is easily shown that

$$P[|y-\bar{y}| \geq k\sigma] \cong \frac{e^{-\frac{1}{2}k^2}}{k\sqrt{2\pi}}, \quad k \gg 1 \tag{12.57}$$

This is a very small probability. For instance for $k = 3$, the probability is only 0.002. Eq. (12.51) will only say that the probability is less than 0.1111, while Eq. (12.56) will say that the probability is less than 0.0493. The difference of these estimates is of course the different degree of information available to the estimation. The more general the assumption, the broader the estimate.

<u>Frequency of Exceeding a Specified Value</u>

For stationary random process, the correlation function $R(\tau)$ can be obtained by time average only. Thus

$$R(\tau) = \widetilde{y(t)y(t+\tau)}$$

Thus the derivative of $R(\tau)$ with respect to τ is

$$\left.\begin{aligned} R'(\tau) &= \widetilde{y(t)y'(t+\tau)} = \widetilde{y(t-\tau)y'(t)} \\ \text{and} \quad R'(\tau) &= -\widetilde{y'(t-\tau)y(t)} = -\widetilde{y'(t)y(t+\tau)} \end{aligned}\right\} \tag{12.58}$$

Hence

$$R'(0) = \widetilde{y(t)y'(t)} = -\widetilde{y'(t)y(t)}$$

Or

$$R'(0) = 0 = \widetilde{yy'} \tag{12.59}$$

Similarly

$$R''(\tau) = \widetilde{y(t)y''(t+\tau)} = -\widetilde{y'(t-\tau)y'(t)}$$

Thus

$$R''(0) = \widetilde{yy''} = -\widetilde{y'^2} \tag{12.60}$$

From Eq. (12.25),

$$R''(0) = -\int_0^\infty \omega^2 f(\omega)\, d\omega \tag{12.61}$$

If both y and y' have a Gaussian probability distribution, then it follows from Eq. (12.44) that they are also statistically independent. The joint probability is then the product of the distribution of y and y'

$$\left.\begin{aligned} W_1(y) &= \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(y-\bar{y})^2}{2\sigma^2}} \\ W_1(y') &= \frac{1}{\sqrt{2\pi\sigma'^2}}\, e^{-\frac{y'^2}{2\sigma'^2}} \end{aligned}\right\} \tag{12.62}$$

and

$$W(y, y') = W_1(y)\, W_1(y') \tag{12.63}$$

In Eq. (12.47), σ and σ' are the mean deviations for y and y' from $\bar{y}$ and $\bar{y}'$ respectively.

For fatigue considerations it will also be useful to ask for the number of times per unit time a certain value is exceeded. Let this number be $N_o(\xi)$, i.e., the number a random function y(t) passes through the value of ξ. $N_o(\xi)$ can be expressed in terms of the joint probability W(y, y'),

$$N_o(\xi) = \int |y'| W(\xi, y') dy' \tag{12.64}$$

This equation is derived from the fact that y spends the time $W_1(y, y')dy\, dy'$ per unit time between y and y + dy having a slope in the range y' and y' + dy'. The crossing of the interval dy takes the time $dy/|y'|$. Hence the expected number of crossings per unit time is $|y'| W(y, y')dy'$. Thus for a double Gaussian distribution, Eqs. (12.47) to (12.49) give

$$N_o(\xi) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\xi-\bar{y})^2}{2\sigma^2}} \int_{-\infty}^{\infty} |y'| \frac{1}{\sqrt{2\pi\sigma'^2}} e^{-\frac{y'^2}{2\sigma'^2}} dy'$$

$$= \frac{1}{2\pi}\frac{\sigma'}{\sigma} e^{-\frac{(\xi-\bar{y})^2}{2\sigma^2}} 2\int_0^{\infty} e^{-\zeta} d\zeta = \frac{1}{\pi}\frac{\sigma'}{\sigma} e^{-\frac{(\xi-\bar{y})^2}{2\sigma^2}} \tag{12.65}$$

The average number of times per unit time that y(t) exceeds ξ is given by $\frac{1}{2} N_o(\xi)$.

The ratio σ'/σ can be calculated from the power spectrum $f(\omega)$: If the $f(\omega)$ is "normalized" by taking out the d.c. component, then

$$\sigma'^2 = \widetilde{y'^2} = \int_0^\infty \omega^2 f(\omega)\, d\omega$$

$$\sigma^2 = \widetilde{y^2} = \int_0^\infty f(\omega)\, d\omega$$

Therefore

$$\frac{\sigma'}{\sigma} = \left[\int_0^\infty \omega^2 f(\omega)\, d\omega \Big/ \int_0^\infty f(\omega)\, d\omega \right]^{1/2} \tag{12.66}$$

Response of a Linear System under Steady Random Input

Let the characteristics of the linear system be represented by its transfer function F(p). The output is y_o and the input is y_i. Then if the power spectrum of y_i is $f(\omega)$ and the correlation function R_i,

$$\overline{y_i^2} = \int_0^\infty f(\omega)\, d\omega = R_i(0) \tag{12.67}$$

and

$$R_i(\tau) = \int_0^\infty f(\omega) \cos \omega\tau\, d\omega = \frac{1}{2} \int_{-\infty}^{\infty} f(\omega)\, e^{i\omega\tau}\, d\omega \tag{12.68}$$

The last relation is possible due to the fact that $f(\omega) = f(-\omega)$, from

Eq. (12.26). Let the power spectrum of y_0 be $g(\omega)$ and the correction function be R_0. Then

$$\overline{y_0^2} = \int_0^\infty g(\omega)\,d\omega = R_0(0) \tag{12.69}$$

$$R_0(\tau) = \int_0^\infty g(\omega)\cos\omega\tau\,d\tau = \frac{1}{2}\int_{-\infty}^{\infty} g(\omega)e^{i\omega\tau}d\omega \tag{12.70}$$

Call h(t) the response of the linear system to a unit impulse at t = 0. For a process going on since $t = -\infty$ and for a system with positive damping, the solution $y_0(t)$ can be written

$$y_0(t) = \int_{-\infty}^{t} y_i(\tau)\,h(t-\tau)\,d\tau$$

Since $h(t) \equiv 0$ for $t < 0$, the upper limit of the integral can be extended to $+\infty$. Then

$$y_0(t) = \int_{-\infty}^{\infty} y_i(\tau)\,h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} y_i(t-s)\,h(s)\,ds$$

The correlation function $R_0(\tau)$ is thus

$$R_0(\tau) = \overline{y_0(t)\,y_0(t+\tau)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{y_i(t-s)\,y_i(t+\tau-s')}\,h(s)\,h(s')\,ds\,ds'$$

But

$$\overline{y_i(t-s)\,y_i(t+\xi-s')} = \overline{y_i(t)\,y_i(t+\tau+s-s')} = R_i(\tau+s-s')$$

By using Eqs. (12.53) and (12.55),

$$\int_{-\infty}^{\infty} g(\omega) e^{i\omega\tau} d\omega = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\omega)\, e^{i\omega(\tau+s-s')} h(s)\, h(s')\, ds\, ds'\, d\omega$$

But for damped systems,

$$F(i\omega) = \int_{-\infty}^{\infty} h(t)\, e^{i\omega t}\, dt$$

Therefore

$$\int_{-\infty}^{\infty} g(\omega) e^{i\omega\tau} d\omega = \int_{-\infty}^{\infty} f(\omega) F(i\omega) F(-i\omega)\, e^{i\omega\tau} d\omega$$

Hence the power spectra $g(\omega)$ and $f(\omega)$ are related as

$$g(\omega) = f(\omega) F(i\omega) F(-i\omega) \qquad (12.71)$$

Thus the character of the output, itself a random function, can be predicted from the input. This problem was first considered perhaps by Langevin in connection with the theory of Browni an motion.

Second Order System

As a simple example, consider the linear system to be of <u>second order</u>, then the equation of motion is one of Langevin's type,

$$m \frac{d^2 y_o}{dt^2} + c\, \frac{dy_o}{dt} + k y_o = y_i \qquad (12.72)$$

The transfer function F(p) of the system is thus

$$F(p) = \frac{1}{mp^2 + cp + k} = \frac{1}{k}\;\frac{1}{\frac{p^2}{\omega_0^2} + 2\zeta\frac{p}{\omega_0} + 1}$$

where ω_0 is the natural frequency of the undamped system and ζ is the ratio of actual damping to the critical damping. Therefore

$$F(i\omega)F(-i\omega) = \frac{1}{k^2\left[\left\{\left(\frac{\omega}{\omega_0}\right)^2 - 1\right\}^2 + 4\zeta^2\left(\frac{\omega}{\omega_0}\right)^2\right]}$$

The power spectrum of the output is thus

$$g(\omega) = \frac{f(\omega)}{k^2\left[\left\{\left(\frac{\omega}{\omega_0}\right)^2 - 1\right\}^2 + 4\zeta^2\left(\frac{\omega}{\omega_0}\right)^2\right]} \qquad (12.73)$$

If we are interested in the mean square output of the linear system, then Eq. (12.54) gives

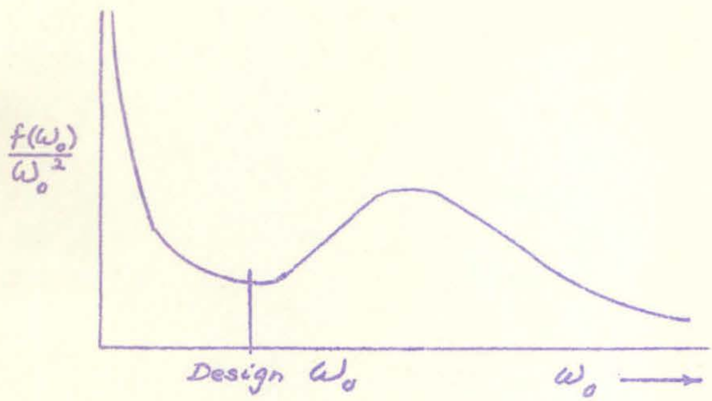
$$\overline{y_0^2} = \frac{1}{k^2}\int_0^\infty \frac{f(\omega)\,d\omega}{\left\{\left(\frac{\omega}{\omega_0}\right)^2 - 1\right\}^2 + 4\zeta^2\left(\frac{\omega}{\omega_0}\right)^2} \qquad (12.74)$$

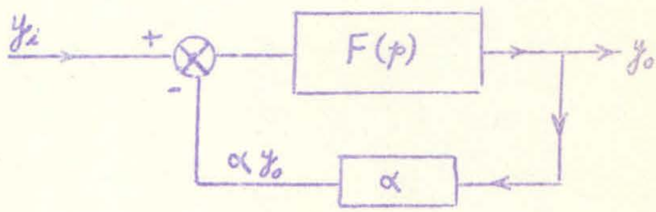
Now if ζ is very small, the denominator of the integrand in Eq. (12.74) is very nearly zero at $\omega = \omega_0$. Therefore if $f(\omega)$ is a slowly varying function, then

$$\overline{y_0^2} = \frac{\omega_0 f(\omega_0)}{k^2}\int_0^\infty \frac{dx}{(x^2-1)^2+4\zeta^2x^2} = \frac{1}{k^2}\omega_0 f(\omega_0)\frac{\pi}{4\zeta}$$

$$= \frac{\pi}{2mc}\,\frac{f(\omega_0)}{\omega_0^2} \qquad (12.75)$$

It is quite conceivable that the function $f(\omega_0)/\omega_0^2$ has a minimum at a reasonable frequency ω_0, then it may be possible to reduce the output radom fluctuation by making the system to operate effectively at ω_0. This can be accomplished by a simple proportional servo-feedback.





Then instead of Eq. (12.57) we have

$$m\frac{d^2y_0}{dt^2} + c\frac{dy_0}{dt} + ky_0 = y_i - \alpha y_0$$

Or

$$m\frac{d^2y_0}{dt^2} + c\frac{dy_0}{dt} + (k+\alpha)y_0 = y_i$$

Then $\omega_0^2 = (k+\alpha)/m$

Therefore by proper choice of α , the output can be reduced. This shows the power and flexibility of servo-control.

Motion of a Two-Dimensional Airfoil in Incompressible Turbulent Flow*

Consider a flat thin airfoil of chord c moving with a constant velocity U through turbulent air. Let x lie along the chord, y along the span of the wing, and z normal to the span and chord. The components of the fluctuating turbulent velocity u, v, w are assumed to be small in comparison to U. Because of these turbulent fluctuations, a time-dependent apparent angle of attack α exists at the airfoil, and, hence, fluxtuating lift forces one produced. The fluctuating angle of attack α is given by

$$\alpha = w/U$$

as long as the fluctuations are small. $\alpha(t)$ now plays the role of the forcing function. The "response" is the fluctuating lift of the airfoil, or better. the lift coefficient $C_L(t)$.

To find the mean square $\overline{C_L^2(t)}$ of the lift coefficient, it is first necessary to define a transfer function for the airfoil. For this purpose, the result of W. R. Sears for airfoil in sinusoidal gust can be utilized. Sears considers the fluctuation of the velocity of the form

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* See H. W. Liepmann: "On the Application of Statistical Concepts to the Buffeting Problem" J. Aero. Sc. 19:793-801 (1952).

$$w(x,t)/U = \alpha_0 e^{i\omega[t - x/U]} \tag{12.76}$$

that is, the vertical velocity consists of harmonic waves of angular frequency and wave length $2\pi U/$. This corresponds to a fixed pattern of sinusoidal gust passing the airfoil at a velocity U, the flight velocity of the wing. For this velocity distribution, the lift coefficient is

$$C_L(t) = 2\pi\alpha_0 e^{i\omega t} \varphi(k) \tag{12.77}$$

where $k = \omega c/2U, \quad c = \text{chord}$

$$\varphi(k) = \frac{J_0(k)K_1(ik) + i J_1(k)K_0(ik)}{K_1(ik) + K_0(ik)} \tag{12.78}$$

J and K are Bessel functions.

Turbulent fluctuations are essentially three dimensional - that is, u, v, w will be functions of x, y, z;t. For the first analysis, it seems sufficient to consider only the component w and the dependence upon x, t. Thus, in turbulent flow we consider a fluctuating velocity or angle of attack of the form

$$\alpha(x,t) = w(x,t)/U$$

If it is now assumed that the turbulence pattern does not change appreciably during a time of the order C/U, then the turbulent angle of attack will also depend upon t-(x, U) only and Sear's result can be applied. This assumption is frequently made in turbulence analysis and requires essentially the condition that the rate of change of the velocity following the particle is small compared with the rate of change of velocity at a fixed position.

With this assumption,

$$\overline{C_L^2} = 4\pi^2 \int_0^\infty f(\omega) \left|\varphi(k)\right|^2 d\omega \tag{12.79}$$

where $f(\omega)$ is the power spectrum of the w/U. According to Eq. (12.32),

$$f(\omega) = \frac{\overline{w^2}}{U^2} \frac{L}{\pi U} \frac{1 + 3\left(L^2\omega^2/U^2\right)}{\left[1 + \left(L^2\omega^2/U^2\right)\right]^2}$$

$\left|\varphi(k)\right|^2$ can be approximated by

$$\left|\varphi(k)\right|^2 \cong \frac{1}{1 + 2\pi k} \tag{12.80}$$

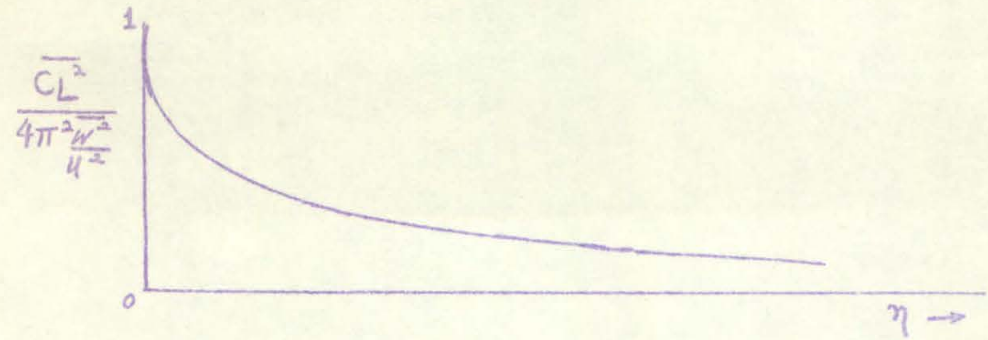
Thus

$$\overline{C_L^2} = 4\pi \frac{\overline{w^2}}{U^2} \int_0^\infty \frac{1 + 3\xi^2}{(1 + \xi^2)^2} \frac{1}{1 + \eta\xi} d\xi$$

$$= 4\pi^2 \frac{\overline{w^2}}{U^2} \left[\frac{4\eta - \pi}{4\pi(\eta^2 + 1)} + \frac{\eta^2 + 3}{2\pi(\eta^2 + 1)^2} \left(\eta \log \eta^2 + \pi\right)\right] \tag{12.81}$$

where

$$\eta = \frac{\pi c}{U} \tag{12.82}$$



Evidently, if $C/L \rightarrow 0$, we have an airfoil of small chord in large-scale turbulence and

$$\overline{C_L^2} \rightarrow 4\pi^2 \frac{\overline{w^2}}{U^2} = 4\pi^2 \overline{\alpha^2}$$

The airfoil behaves quasi-stationary with a lift slope of 2π. If, on the other hand, C/L becomes large, we deal with an airfoil of long chord in small-scale turbulence. It follows from Eq. (12.66), that $\overline{C_L^2} \rightarrow 0$. That is to say the "gusts" cancel each other out completely and the net lift is zero as might be expected.

Intermittent Forcing Function

A phenomenon of major importance for aerodynamic buffeting is the so-called "intermittency" in wake flow. That is, the edge of any wake fluctuates with a large-scale motion so that a point situated near the edge will sometimes be within the wake and sometimes outside the wake. For a tail surface situated near the edge of the wake of a stalled or partially stalled wing, this "intermittency" may thus be extremely important in determinig the lift forces. A detailed analysis would require much more experimental study than is available at present. For a crude idea of the effect, consider the flow at the tail as a region of uniform downwash switched on and off at irregular intervals. Such a flow is probably a good model for the conditions in the wake of an intermittently stalling wing. If the probability of switching over from one region to the other is assumed to be governed by a Poisson distribution, one can then apply the power spectrum of Eq. (12.49) with some modifications: The mean deviation is not unity but the mean angle $\sqrt{\overline{w^2}/U^2}$. The average time T for the forcing function be switched on is the mean interval $1/\beta$. Thus the power spectrum is

$$f(\omega) = \frac{\overline{w^2}}{U^2}\frac{T}{\pi}\frac{1}{1+\left(\frac{\omega T}{2}\right)^2} \tag{12.43}$$

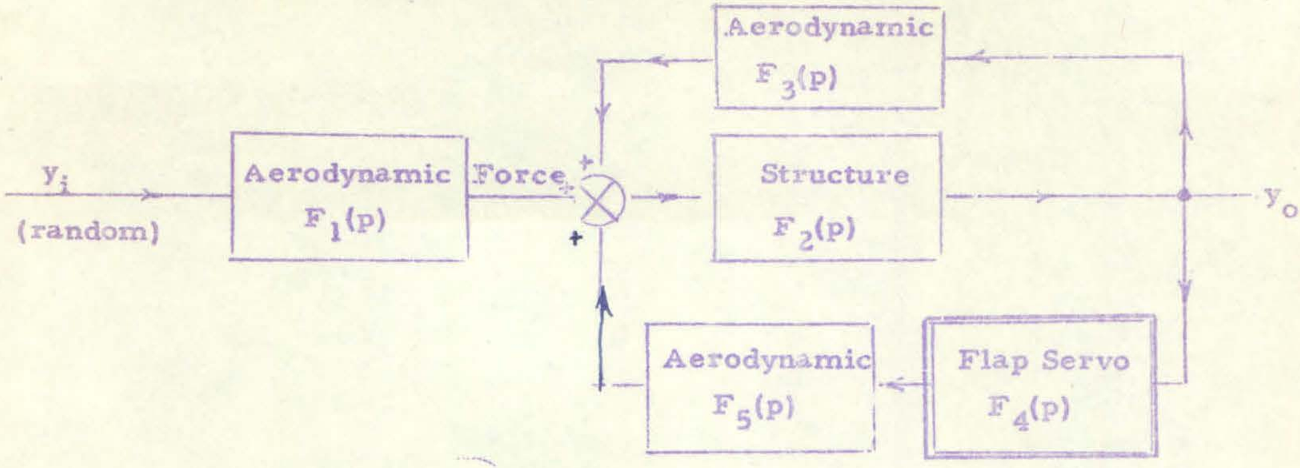
Then the mean square lift coefficient is approximately

$$\overline{C_L^2} = \frac{\overline{w^2}}{U^2}\frac{T}{\pi}4\pi^2\int_0^\infty \frac{d\omega}{\left[1+\left(\frac{\omega T}{2}\right)^2\right]\left[1+\pi\frac{\omega c}{U}\right]}$$

$$= 4\pi^2\frac{\overline{w^2}}{U^2}\frac{2}{\pi}\frac{\eta\log\eta+\frac{\pi}{2}}{1+\eta^2}\,, \qquad \eta = \frac{2\pi c}{UT} \tag{12.84}$$

Servo-Design for Random Forcing Function

We have already indicated in connection with the response of a second order system to random input the possibility of servo-control. In that instance, however, the feedback mechanism is rather primitive in that the feedback control force required is of the same order of magnitude as the input forcing function. In more practical design, this feedback mechanism can be made much more subtle, so as to reduce the control force required. For instance, a wing in a turbulent flow can be controlled by a feedback servo which moves the hinged flap. The force necessary to move the flap could be quite small in comparison with aerodynamic effects such movement produces. The servo link can be thought of as follows: The input random function is the turbulent air stream. The first aerodynamic transfer function $F_1(p)$ is the relation between the turbulent instream and the aerodynamic lift due to the turbulent airstream. As a result of the changing lift and moment, the wing is subject to vertical and torsional motion. The relation between the aerodynamic forces and wing motion is given by the structural transfer function $F_2(p)$. The wing motion will have two effects. There is the aerodynamic forces due to wing motion through the second aerodynamic transfer function $F_3(p)$. This is the first feedback loop. This loop is however not under designer's control. The designer's control is on the second loop. The wing motion can be used to generate flap motion through the transfer function $F_4(p)$. The flap motion will again generate aerodynamic forces through the transfer function $F_5(p)$. The loop diagram is thus as follows:



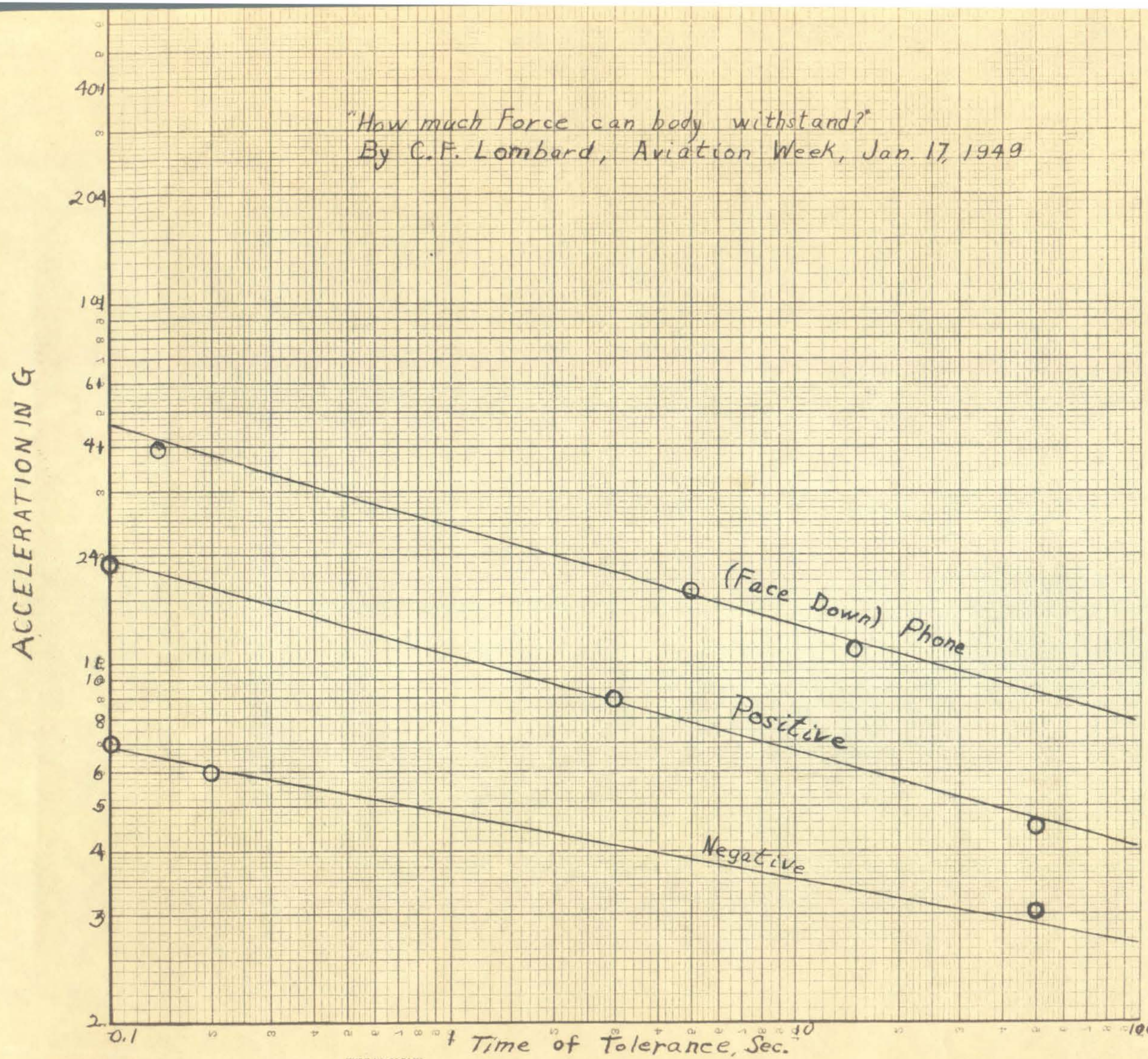
Thus the input and output relation is as follows:

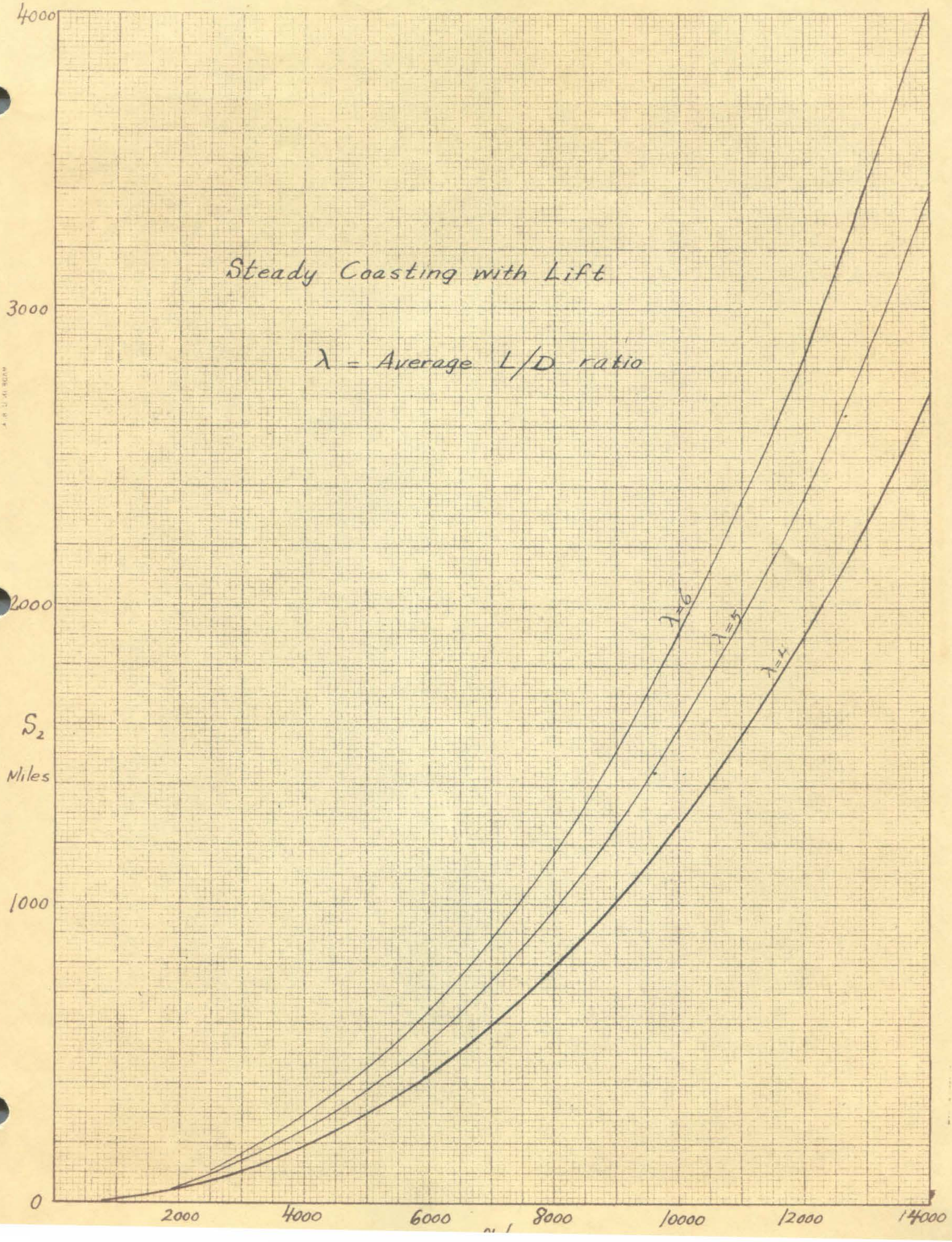
$$Y_o = F_2(p)\left[F_1(p)Y_i + F_3(p)Y_o + F_5(p)F_4(p)Y_o\right]$$

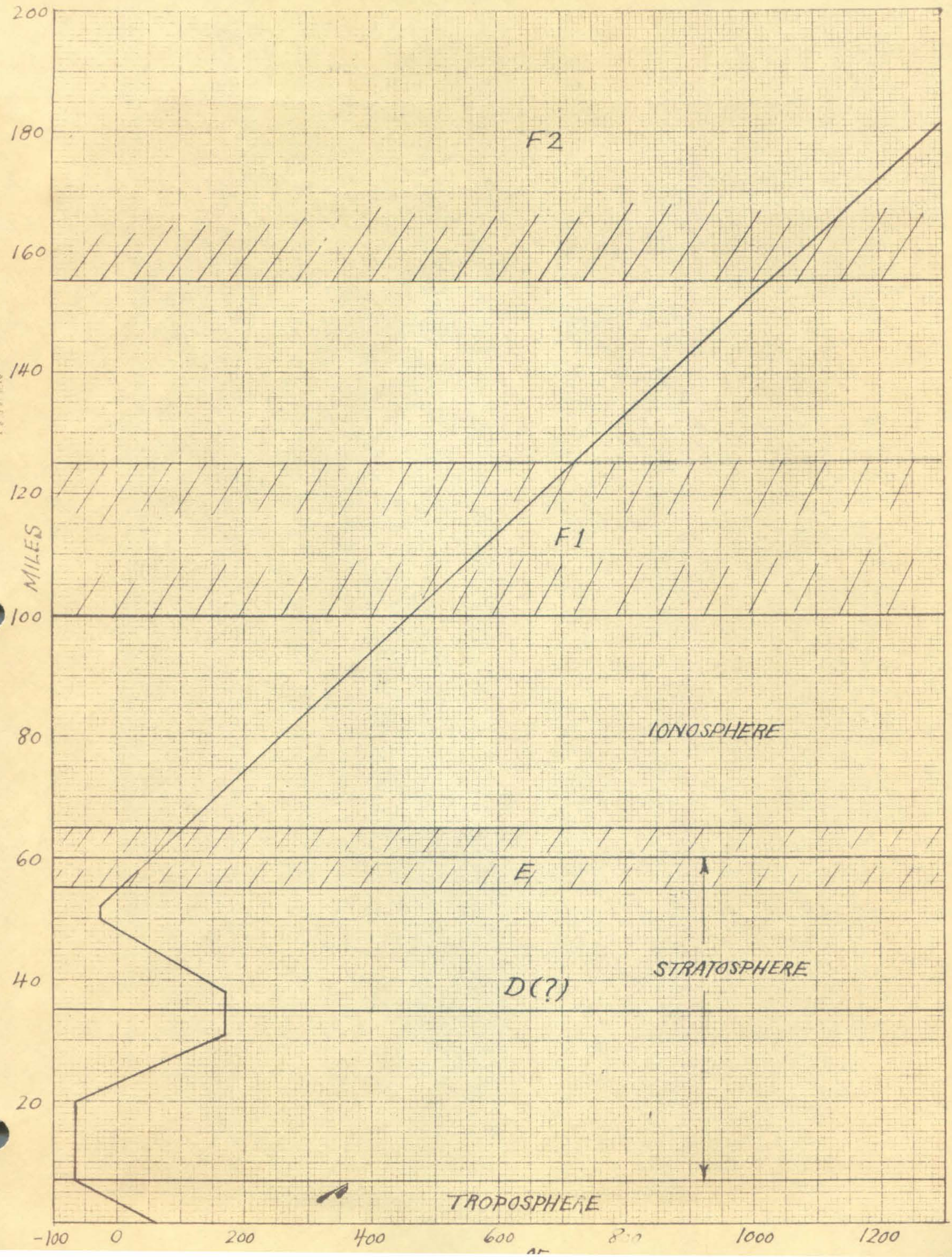
Or

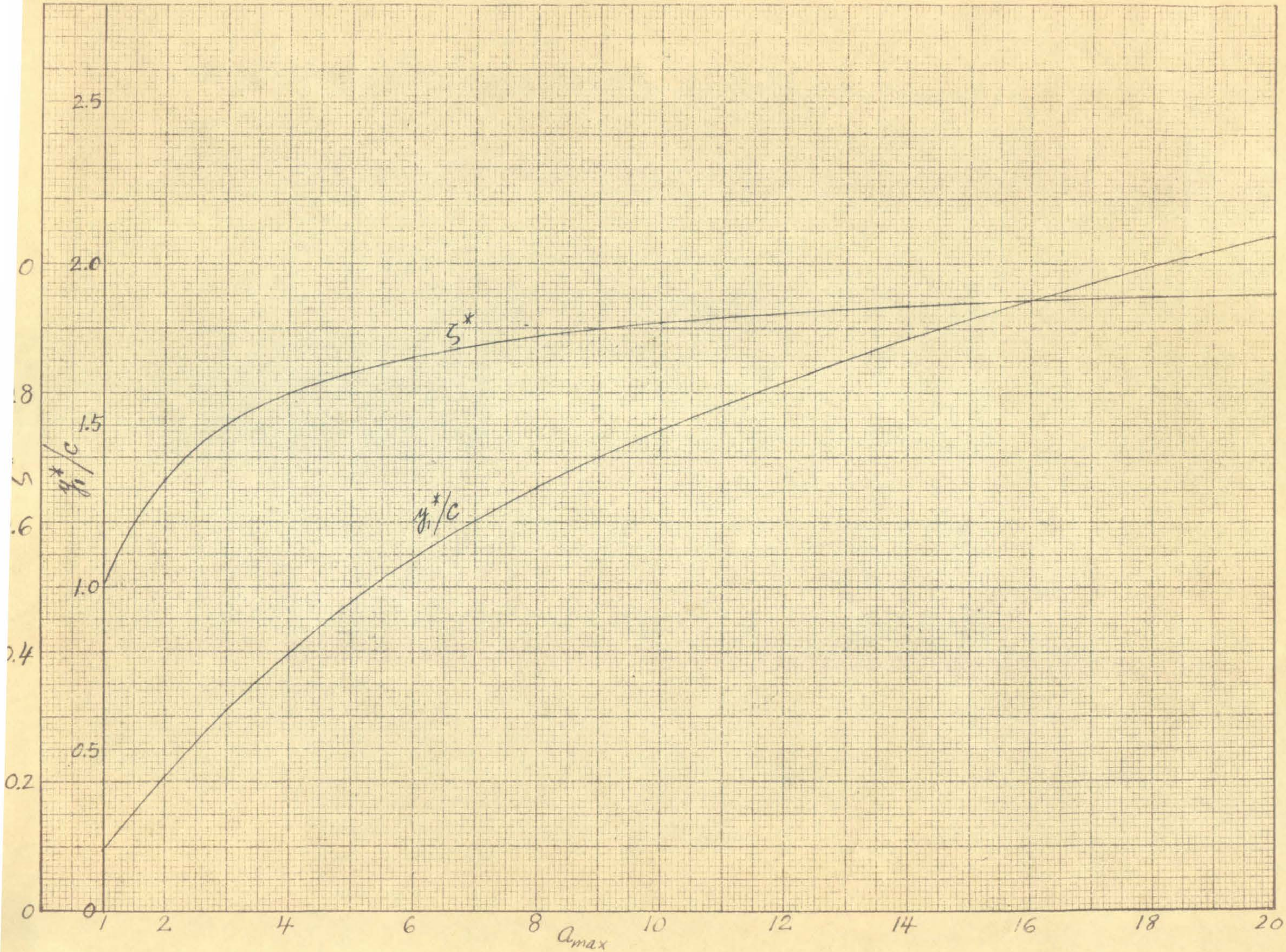
$$\frac{Y_o}{Y_i} = F(p) = \frac{F_1(p)F_2(p)}{1 - F_2(p)\left[F_3(p) + F_5(p)F_4(p)\right]} \tag{12.85}$$

The overall transfer function $F(p)$ can thus be modified by changing the servo transfer function $F_4(p)$. The design aim would then be the maximum reduction of the wing motion y_o without excessive flap motion by properly designing the servo.

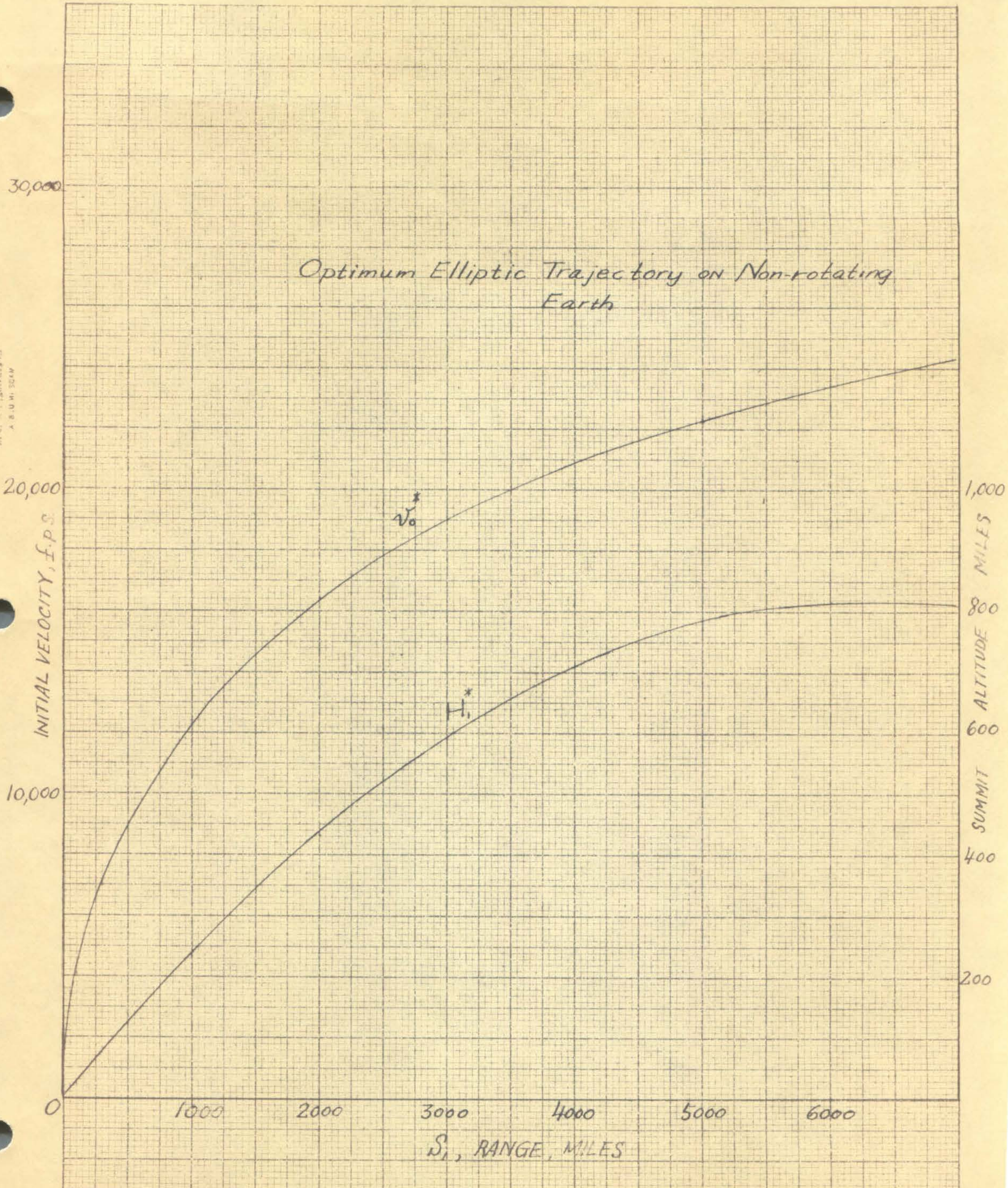

"How much Force can body withstand?"
By C.F. Lombard, Aviation Week, Jan. 17, 1949
ACCELERATION IN G
Time of Tolerance, Sec.
(Face Down) Phone
Positive
Negative
0.1
10
100
2
3
4
5
6
8
10
20
40
60
100
200
400

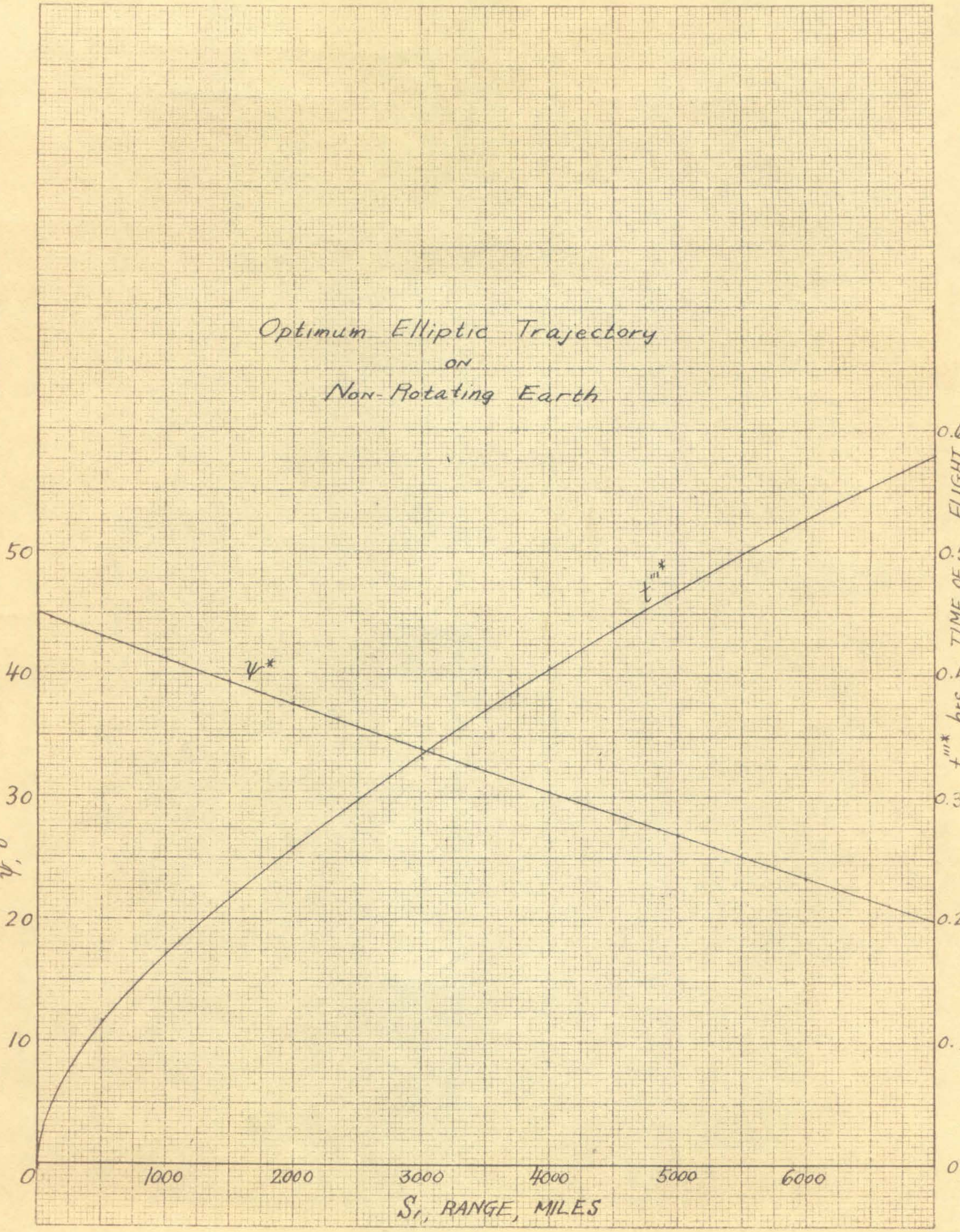

Steady Coasting with Lift
λ = Average L/D ratio
λ=6
λ=5
λ=4
4000
3000
2000
1000
0
S₂
Miles
2000
4000
6000
8000
10000
12000
14000


200
180
160
140
120
100
80
60
40
20
MILES
F2
F1
IONOSPHERE
E
STRATOSPHERE
D(?)
TROPOSPHERE
-100
0
200
400
600
1000
1200

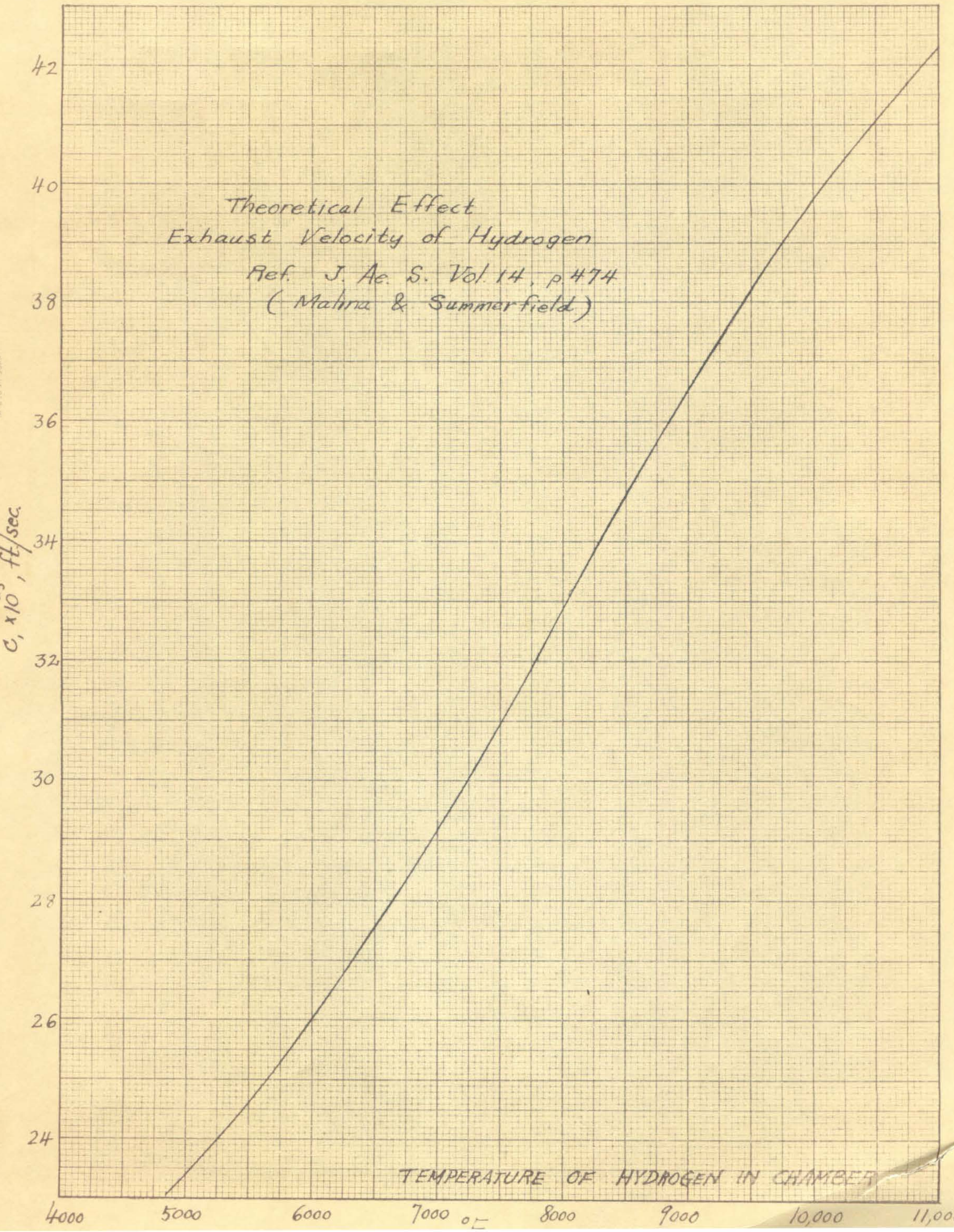


2.5
2.0
1.5
1.0
0.5
0
0
.8
.6
.4
0.2
0
ζ*
y*₁/c
y*₁/c
1
2
4
6
8
10
12
14
16
18
20
a_max


Optimum Elliptic Trajectory on Non-rotating Earth
30,000
20,000
10,000
0
INITIAL VELOCITY, f.p.s.
1,000
800
600
400
200
SUMMIT ALTITUDE MILES
v_0^*
H_1^*
1000
2000
3000
4000
5000
6000
S_1, RANGE, MILES



Optimum Elliptic Trajectory
on
Non-Rotating Earth
ψ^*
t'''^*
$\psi, °$
10
20
30
40
50
0
1000
2000
3000
4000
5000
6000
S_r, RANGE, MILES
0
0.1
0.2
0.3
0.4
0.5
0.6
t'''^*, hrs. TIME OF FLIGHT


Theoretical Effect
Exhaust Velocity of Hydrogen
Ref. J. Ae. S. Vol. 14, p. 474
(Malina & Summerfield)
c, x10^-3, ft/sec
42
40
38
36
34
32
30
28
26
24
TEMPERATURE OF HYDROGEN IN CHAMBER
4000
5000
6000
7000
°F
8000
9000
10,000
11,000